SOME REMARKS TO THE FORMAL AND LOCAL THEORY OF THE GENERALIZED DHOMBRES FUNCTIONAL EQUATION

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Abstract. We are looking for local analytic respectively formal solutions of the generalized Dhombres functional equation $f(zf(z)) = \varphi(f(z))$ in the complex domain. First we give two proofs of the existence theorem about solutions f with $f(0) = w_0$ and $w_0 \in \mathbb{C}^\star \setminus \mathbb{E}$ where $\mathbb{E}$ denotes the group of complex roots of 1. Afterwards we represent solutions f by means of infinite products where we use on the one hand the canonical convergence of complex analysis, on the other hand we show how solutions converge with respect to the weak topology. In this section we also study solutions where the initial value z_0 is different from zero.

1. Introduction

We study the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

in the complex domain. The function φ is known and we are looking for local analytic or formal solutions f of (1.1). The original Dhombres equation, which was inaugurated by Jean Dhombres in the real domain, see [2], is given by

$$f(xf(x)) = f(x)^2.$$

In the complex domain equation (1.1) was introduced by L. Reich, J. Smítal and M. Štefánková in [3]. They were looking for non constant solutions f of (1.1) with $f(0) = 0$. In a subsequent paper, namely in [4], the authors discussed solutions f with $f(0) = w_0$ where w_0 is a complex number different from zero and also no complex root of unity. Therefore it is used that if $f(0) = w_0$ and f is non constant a function g with $g(z) = c_k z^k + c_{k+1} z^{k+1} + \dots$ and $k \in \mathbb{N}$, $c_k \neq 0$ can be established such that $f(z) = w_0 + g(z)$. Substituting this representation of f in (1.1) leads to the transformed generalized Dhombres functional equation

$$g(w_0 z + zg(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

where the function $\tilde{\varphi}$ is computed by $\varphi(y) = w_0 + \tilde{\varphi}(y - w_0)$ and hence it is given by $\tilde{\varphi}(y) = w_0^k y + d_2 y^2 + \dots$ which can be shown by comparing the coefficients of (1.2). In [4] Proposition 2 on page 824 is proved. We will denote Proposition 2 by Theorem 1.1. By Γ_1 we denote the set of all formal series beginning with z.

Theorem 1.1. *Let w_0 be a complex number different from zero and no root of unity, and let the function $\tilde{\varphi}$ be given by $\tilde{\varphi}(y) = w_0^k y^k + \dots$ for a $k \in \mathbb{N}$. Then there exists a unique function $\tilde{g}_0 \in \Gamma_1$ such that the set of non constant soutions g of*

$$g(w_0 z + zg(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

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in $\mathbb{C}[\![z]\!]$ *is given by*

$$\mathcal{L}_{\tilde{\varphi}} = \left\{ g | g(z) = \tilde{g}_0(c_k z^k), c_k \in \mathbb{C}^\star \right\}.$$

Then the set $\mathcal{L}_{\tilde{\varphi}}$ *is a subset of* $\mathbb{C}[\![z^k]\!]$. *Furthermore equation* (1.2) *has a unique solution* g *such that* $g(z) = c_k z^k + \ldots$, *for every* $c_k \in \mathbb{C}^\star$.

In the following section we want to give two alternative proofs of Theorem 1.1 like it is suggested in [4]. In the third section we investigate solutions of the generalized Dhombres functional equation where infinite products are involved. There we also investigate solutions with initial value $z_0 \neq 0$, that means solutions f of (1.1) where $f(z_0) = w_0$ and $z_0 \neq 0$.
Before we start, we want to explain our notations, we use the same notations which are used in [4], and therefore we give the following definition. An introduction to the ring of formal power series can be found in the book of H. Cartan [1].

Definition 1.2. By

$$\mathbb{C}[\![z]\!] = \left\{ F | F(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \ldots \right\}$$

with $\beta_\nu \in \mathbb{C}$ for $\nu \geq 0$ we denote the set of formal power series. For a series $F \in \mathbb{C}[\![z]\!]$ the order of F is defined by $ord\, F := min\{\nu \in \mathbb{N} | \beta_\nu \neq 0\}$ and one sets ∞ for the order of the trivial series. For $F, G \in \mathbb{C}[\![z]\!]$ we use $F(z) \equiv G(z) \pmod{z^l}$ if $ord\,(F - G) \geq l$. We have

$$\mathbb{C}[\![z^k]\!] = \{F | F \in \mathbb{C}[\![z]\!], F(z) = \sum_{\nu \geq 0} \beta_\nu z^\nu, \beta_\nu = 0 \text{ if } \nu \not\equiv 0 \pmod{k}\}$$

for $k \in \mathbb{N}$. A series $F \in z\mathbb{C}[\![z^k]\!]$ has the representation $F(z) = \beta_1 z + \beta_{k+1} z^{k+1} + \beta_{2k+1} z^{2k+1} + \ldots$. We also need $\Gamma = \{F | F \in \mathbb{C}[\![z]\!] \text{ and } ord\ F = 1\}$, and $(\Gamma, \circ)$ is the group of invertible formal power series with respect to substitution $\circ$ in $\mathbb{C}[\![z]\!]$. Then the set $\Gamma_1 = \left\{F | F \in \Gamma \text{ and } F(z) \equiv z \pmod{z^2}\right\}$ is a subgroup of $(\Gamma, \circ)$. Finally we define

$$\mathbb{C}^{(k)}[\![z]\!] = \{F | F \in \mathbb{C}[\![z]\!] \text{ with } F(z) = \sum_{\nu \geq 0} c_\nu z^\nu \text{ where } c_\nu = 0 \text{ if } \nu \not\equiv 1 \pmod{k}\}$$

which leads to $\Gamma^{(k)} = \Gamma \cap \mathbb{C}^{(k)}[\![z]\!]$ and $\Gamma_1^{(k)} = \Gamma_1 \cap \mathbb{C}^{(k)}[\![z]\!]$. The set $\mathcal{U}_{\mathbb{C}[\![z]\!]}$ defined as $\mathcal{U}_{\mathbb{C}[\![z]\!]} = \{F | F \in \mathbb{C}[\![z]\!] \text{ und } F(z) \equiv 1 \pmod{z}\}$ is a subgroup of all multiplicative units in $\mathbb{C}[\![z]\!]$. By $\mathbb{C}^\star$ we denote the set $\mathbb{C} \setminus \{0\}$.

2. Alternative Proofs

Before we start with our first proof of Theorem 1.1 we have to provide the following theorem which deals with linearizations. For the definition of a Siegel number we refer the reader to [5].

Theorem 2.1. *Let the function* $\phi \in \Gamma$, $\phi(z) = \lambda z + \xi_2 z^2 + \ldots$ *be given.*

(1) *If* $\lambda \in \mathbb{C}^\star \setminus \mathbb{E}$ *then there exists exactly one solution* $S_\phi \in \Gamma_1$ *of the linearization equation*

$$\phi(S(z)) = S(\lambda z). \tag{2.1}$$

(2) *If the function* ϕ *is local analytic in a sufficiently small neighbourhood of zero and if* $|\lambda| \neq 1$ *or* λ *is a Siegel number, then also* S_ϕ *is local analytic in a sufficiently small neighbourhood of zero.*
(3) *If* $\phi \in \Gamma^{(k)}$ *then for every solution* S_ϕ *we have* $S_\phi \in \Gamma^{(k)}$.

The proof of the first and second part of Theorem 2.1 can be found in [5] on the pages 157 - 174. The third part is proved in [4] on page 823.
Now we want to prove Theorem 1.1 and therefore we use the suggestions which are provided in Remark 2 in [4].

First proof of Theorem 1.1. The complex number w_0 is not zero and no root of unity and hence also the complex number w_0^k for a natural number k is no root of unity. According to Theorem 2.1 we consider the linearization equation

$$\tilde{\varphi}(S(z)) = S(w_0^k z).$$

Then there exists a solution $S_{\tilde{\varphi}}$ of (2.1) such that $S_{\tilde{\varphi}} \in \Gamma_1$ and such that

$$\tilde{\varphi}(S_{\tilde{\varphi}}(z)) = S_{\tilde{\varphi}}(w_0^k z)$$

holds. In this equation we substitute $S_{\tilde{\varphi}}^{-1}(z)$ for z and this substitution leads to the equivalent expression

$$\tilde{\varphi}(z) = S_{\tilde{\varphi}}(w_0^k S_{\tilde{\varphi}}^{-1}(z)).$$

This representation of $\tilde{\varphi}$ is substituted in the transformed generalized Dhombres functional equation (1.2) $g(w_0 z + zg(z)) = \tilde{\varphi}(g(z))$. Then we get

$$g(w_0 z + zg(z)) = S_{\tilde{\varphi}}(w_0^k S_{\tilde{\varphi}}^{-1}(g(z)))$$

which becomes equivalent to

$$S_{\tilde{\varphi}}^{-1}(g(w_0 z + zg(z))) = w_0^k S_{\tilde{\varphi}}^{-1}(g(z)).$$

Then we define the function h as $h := S_{\tilde{\varphi}}^{-1} \circ g$ and hence we obtain

$$h(w_0 z + zg(z)) = w_0^k h(z).$$

Using the representation $g(z) = S_{\tilde{\varphi}}(h(z))$ leads to

$$h(w_0 z + zS_{\tilde{\varphi}}(h(z))) = w_0^k h(z).$$

The function $S_{\tilde{\varphi}}$ is the uniquely determinded solution of (2.1) such that $S_{\tilde{\varphi}}(z) = z + \ldots$. We define the function A as $S_{\tilde{\varphi}}(z) = A(z) = z + \ldots$ and therefore we get

$$h(w_0 z + zA(h(z))) = w_0^k h(z).$$

Next we consider a function R with order equal to one such that $h(z) = R(z)^k$ holds. This expression is substituted in the equation above and this leads to

$$R(w_0 z + zA(R(z)^k)))^k = w_0^k R(z)^k.$$

After taking the $k-th$ root the equation

$$R(w_0 z + zA(R(z)^k)) = \epsilon w_0 R(z)$$

where ϵ has to be determinded, remains. Let the function R be given by $R(z) = r_1 z + r_2 z^2 + \ldots$ with $r_1 \neq 0$, then $R(z)^k = r_1^k z^k + \ldots$ and $zA(R(z)^k) = r_1^k z^{k+1} + \ldots$. A detailed discussion of the left hand side of this equation results in $R(w_0 z + zA(R(z)^k)) = r_1 w_0 z + \ldots$, while the right hand side has the representation $\epsilon w_0 R(z) = \epsilon w_0 r_1 z + \ldots$ where $r_1 \neq 0$ and $w_0 \neq 0$ and hence ϵ has to be equal one. Altogether we have

$$R(w_0 z + zA(R(z)^k)) = w_0 R(z).$$

Next we substitute $R^{-1}(z)$ for z and so we get

$$R(w_0 R^{-1}(z) + R^{-1}(z)A(z^k)) = w_0 z.$$

Defining the function Q as $Q(z) := R^{-1}(z)$ then we know, because we suppose that $r_1 \neq 0$, that $Q(z) = q_1 z + q_2 z^2 + \dots$ and the equation above is equivalent to

$$(w_0 + A(z^k))Q(z) = Q(w_0 z). \tag{2.2}$$

In a next step we show that the set of solutions of (2.2) is given by

$$\{Q(z) \mid Q(z) = q_1 Q^0(z),\ q_1 \in \mathbb{C}^\star\}$$

where $Q^0 \in \Gamma_1$ is the uniquely determinded solution of (2.2) and $Q^0 \in \Gamma^{(k)}$. A solution Q of (2.2) is uniquely determinded by $Q(z) \equiv q_1 z \pmod{z^2}$.
We can represent every possible solution $Q(z)$ with $Q(z) \equiv q_1 z \pmod{z^2}$ by $Q(z) = q_1 Q^0(z)$ where $Q^0 \in \Gamma_1$ and Q^0 is a solution of (2.2). Furthermore Q^0 can be written as $Q^0(z) = zQ^*(z)$ where $Q^* \in \mathbb{C}[\![z]\!]$ and $Q^*(z) = 1 + \dots$. We substitute this representation of Q, namely $Q(z) = q_1 z Q^*(z)$ in (2.2) and so we get $(w_0 + A(z^k))q_1 z Q^*(z) = w_0 q_1 z Q^*(w_0 z)$, respectively

$$(1 + w_0^{-1} A(z^k))Q^*(z) = Q^*(w_0 z). \tag{2.3}$$

We know that $Q^*(z) = 1 + \dots$ and also that $(1 + w_0^{-1} A(z^k)) = 1 + \dots$ and therefore it is possible to use the formal logarithm. Let the series F be defined by $F(z) := Ln(1 + w_0^{-1} A(z^k))$ where *ord* $F(z) \geq 1$ and let the series G be defined by $G(z) := Ln(Q^*(z))$ where *ord* $G(z) \geq 1$. After applying the formal logarithm to (2.3) we obtain the equation $F(z) + G(z) = G(w_0 z)$, respectively

$$F(z) = G(w_0 z) - G(z). \tag{2.4}$$

It is known that $F(z) \in \mathbb{C}[\![z^k]\!]$ which follows from the property that $A(z^k) \in \mathbb{C}[\![z^k]\!]$ and the use of the formal logarithm. Next we use the representation of F and G as formal power series and therefore we write $F(z) = \sum_{\nu=1}^{\infty} \alpha_\nu z^\nu$ and $G(z) = \sum_{\nu=1}^{\infty} \gamma_\nu z^\nu$ where of course the coefficients of F are known. Substituting the expressions of F and G in (2.4) yields

$$\begin{aligned} \alpha_1 z + \alpha_2 z^2 + \dots + \alpha_\nu z^\nu + \dots &= \gamma_1 w_0 z + \gamma_2 w_0^2 z^2 + \dots \gamma_\nu w_0^\nu z^\nu + \dots \\ &\quad - \gamma_1 z - \gamma_2 z^2 - \dots - \gamma_\nu z^\nu \\ &= \gamma_1 z(w_0 - 1) + \gamma_2 z^2 (w_0^2 - 1) + \dots \\ &\quad + \gamma_\nu z^\nu (w_0^\nu - 1) + \dots . \end{aligned}$$

After comparing the coefficients we obtain $\alpha_\nu = \gamma_\nu (w_0^\nu - 1)$ for $\nu \geq 1$. The complex number w_0 is no root of unity and hence for $\nu \geq 1$ we get

$$\gamma_\nu = \alpha_\nu (w_0^\nu - 1)^{-1}$$

for the coefficients of G. Therefore the series G is uniquely determinded. Since $F(z) \in \mathbb{C}[\![z^k]\!]$ the coefficients γ_ν equal 0 if k does not divide ν and hence also $G(z) \in \mathbb{C}[\![z^k]\!]$. Reversing our calculations leads to $Q^*(z) = \exp(G(z))$ and as a consequence of the exponential funtion $Q^*(z) \in \mathbb{C}[\![z^k]\!]$. Furthermore we get $Q^0(z) = zQ^*(z) \in \mathbb{C}^{(k)}[\![z]\!] \cap \Gamma_1 = \Gamma_1^{(k)}$, and hence

$$Q(z) = q_1 Q^0(z) \in \Gamma^{(k)},$$

therefore the step is proved.
The function R^{-1} is defined as $R^{-1} = Q$ and so also $R^{-1}(z) \in \Gamma^{(k)}$ respectively $R(z) \in \Gamma^{(k)}$. From our transformations we know that g has the representation $g(z) = (S_{\tilde{\varphi}} \circ R^k)(z)$ and now we want to compute g in detail. The function R is an element of $\Gamma^{(k)}$ and hence we can represent R as $R(z) = r_1 z + r_{2k+1} z^{2k+1} +$

$r_{3k+1}z^{3k+1}+\ldots$. Then we get $R(z)^k = r_1^k z^k + \ldots$ and $S_{\tilde{\varphi}}(R(z)^k) = r_1^k z^k + \ldots$. We obtain

$$g(z) = S_{\tilde{\varphi}}(R(z)^k) = r_1^k z^k + \ldots \in \mathbb{C}[\![z^k]\!].$$

Using the representation $R(z) = r_1 z(1+\sum_{\nu\geq 2}\tilde{r}_\nu (r_1^k z^k)^\nu)$ for the function R where $\tilde{r}_\nu = r_1^{-1} r_{\nu k+1} r_1^{-\nu k}$ for $\nu \geq 2$ we get

$$R(z)^k = r_1^k z^k (1 + \sum_{\nu\geq 2} \tilde{r}_\nu (r_1^k z^k)^\nu)^k.$$

Defining c_k and r_1 as $c_k := r_1^k$ and $r_1 := q_1^{-1}$ leads to

$$g(z) = S_{\tilde{\varphi}}(R(z)^k) = r_1^k z^k (1 + \sum_{\nu\geq 1} \delta_\nu (r_1^k z^k)^\nu) = c_k z^k (1 + \sum_{\nu\geq 1} \delta_\nu (c_k z^k)^\nu)$$

where the coefficient δ_ν for $\nu \geq 1$ is uniquely determinded. Then we define

$$\tilde{g}_0(y) = y(1 + \sum_{\nu\geq 1} \delta_\nu y^\nu) \in \Gamma_1 \subseteq \mathbb{C}[\![y]\!],$$

and so we get

$$g(z) = \tilde{g}_0(c_k z^k) \tag{2.5}$$

with $g(z) \equiv c_k z^k \pmod{z^{k+1}}$ for every solution g of (1.2). It is possible to reverse our calculations and therefore for every $c_k \in \mathbb{C}^\star$ we obtain a solution $g(z)$ of the transformed generalized Dhombres functional equation (1.2) which is defined by (2.5). By comparing coefficients and in particular by the special representation of g in terms of $\tilde{g}_0$ in (2.5) we see that for every $c_k \in \mathbb{C}^\star$ there exists exactly one solution. □

Now we want to give a second proof of Theorem 1.1. Therefore we compute a $\tilde{g}_0$ such that a set of solutions of (1.2) is given by $\{g | g(z) = \tilde{g}_0(c_k z^k), c_k \in \mathbb{C}^\star\}$. Then we show that these are all solutions. The part of the proof where we show that the solutions are unique is adopted from [4] Remark 3.

Second proof of Theorem 1.1. In a first step we are looking for a $\tilde{g}_0 \in \Gamma_1$ such that for every $c_k \in \mathbb{C}^\star$ the function g defined as $g(z) := \tilde{g}_0(c_k z^k)$ is a solution of (1.2). We substitute the representation $g(z) := \tilde{g}_0(c_k z^k)$ in the transformed generalized Dhombres functional equation

$$g(w_0 z + z g(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

and hence we get $\tilde{g}_0(c_k(zw_0 + z\tilde{g}_0(c_k z^k))^k) = \tilde{\varphi}(\tilde{g}_0(c_k z^k))$, respectively

$$\tilde{g}_0(c_k z^k (w_0 + \tilde{g}_0(c_k z^k))^k) = \tilde{\varphi}(\tilde{g}_0(c_k z^k)).$$

After defining $y := c_k z^k$ the equation above becomes equivalent to

$$\tilde{g}_0(y(w_0 + \tilde{g}_0(y))^k) = \tilde{\varphi}(\tilde{g}_0(y)).$$

Substituting $\tilde{g}_0^{-1}(y)$ for y leads to

$$\tilde{g}_0(\tilde{g}_0^{-1}(y)(w_0 + y)^k) = \tilde{\varphi}(y).$$

Now we introduce a function $\tilde{h}_0$ such that $\tilde{h}_0 := \tilde{g}_0^{-1}$ and hence after an additional transformation we have to consider the equation

$$(w_0 + y)^k \tilde{h}_0(y) = \tilde{h}_0(\tilde{\varphi}(y)).$$

Like in the first proof we can according to Theorem 2.1 write $S_{\tilde{\varphi}}(w_0^k S_{\tilde{\varphi}}^{-1}(y))$ instead of $\tilde{\varphi}$ where the linearization function is denoted as $S_{\tilde{\varphi}}$. After substituting this representation of $\tilde{\varphi}$ in the equation above we obtain

$$(w_0+y)^k \tilde{h}_0(y) = \tilde{h}_0(S_{\tilde{\varphi}}(w_0^k S_{\tilde{\varphi}}^{-1}(y))$$

respectively

$$(w_0+y)^k \tilde{h}_0(S_{\tilde{\varphi}}(S_{\tilde{\varphi}}^{-1}(y))) = \tilde{h}_0(S_{\tilde{\varphi}}(w_0^k S_{\tilde{\varphi}}^{-1}(y))).$$

Then we define a function Q such that $Q := \tilde{h}_0 \circ S_{\tilde{\varphi}}$ and so we get

$$(w_0+y)^k Q(S_{\tilde{\varphi}}^{-1}(y)) = Q(w_0^k S_{\tilde{\varphi}}^{-1}(y)).$$

We substitute $S_{\tilde{\varphi}}(y)$ for y and hence we get

$$(w_0 + S_{\tilde{\varphi}}(y))^k Q(y) = Q(w_0^k y). \tag{2.6}$$

To solve (2.6) we use $Q(y) = yQ^*(y)$ where $Q^* \in \mathcal{U}_{\mathbb{C}[\![y]\!]}$ and $Q(y) = yQ^*(y) = y(1+q_2y+\ldots)$ as representation of Q. This representation is substituted in (2.6) and so we get

$$w_0^k(1+w_0^{-1}S_{\tilde{\varphi}}(y))^k yQ^*(y) = w_0^k yQ^*(w_0^k y)$$

which is equivalent to

$$(1+w_0^{-1}S_{\tilde{\varphi}}(y))^k Q^*(y) = Q^*(w_0^k y). \tag{2.7}$$

Then we have $(1+w_0^{-1}S_{\tilde{\varphi}}(y))^k = 1+\ldots$ and $Q^*(y) = 1+\ldots$ and hence we can use the formal logarithm, we define the functions F and G as $F(y) := Ln(1+w_0^{-1}S_{\tilde{\varphi}}(y))^k = kLn(1+w_0^{-1}S_{\tilde{\varphi}}(y))$ where we used the properties of the logarithm, respectively $G(y) := Ln(Q^*(y))$. Therefore (2.6) becomes equivalent to the functional equation

$$F(y) = G(w_0^k y) - G(y). \tag{2.8}$$

The uniquely determinded function $S_{\tilde{\varphi}}$ has the representation $S_{\tilde{\varphi}}(y) = y + s_2y^2 + s_3y^3 + \ldots$, and computing the function $F(y)$ in detail leads to

$$\begin{aligned} F(y) &= kLn(1+w_0^{-1}S_{\tilde{\varphi}}(y)) \\ &= kw_0^{-1}S_{\tilde{\varphi}}(y) - \frac{kw_0^{-2}S_{\tilde{\varphi}}(y)^2}{2} + \frac{kw_0^{-3}S_{\tilde{\varphi}}(y)^3}{3} - \ldots \\ &= kw_0^{-1}y + kw_0^{-1}s_2y^2 + kw_0^{-1}s_3y^3 + \ldots - \frac{kw_0^{-2}y^2}{2} - \ldots \in \mathbb{C}[\![y]\!]. \end{aligned}$$

Therefore also the function G is element of $\mathbb{C}[\![y]\!]$ and, because w_0 is no root of unity, we can uniquely determine the coefficients of G from the functional equation (2.8). Then it follows that $Q^*(y) \in \mathbb{C}[\![y]\!]$ and hence $Q(y) \in \Gamma_1^{(1)}$ is uniquely determinded. Therefore the same is true for $\tilde{h}_0 = Q \circ S_{\tilde{\varphi}}^{-1}$ and finally the function $\tilde{g}_0$ is uniquely determinded because we used the transformation $\tilde{g}_0^{-1} = \tilde{h}_0$. Summarizing this, results in $\tilde{g}_0(y) \in \mathbb{C}[\![y]\!]$, where $\tilde{g}_0(y) = y + \ldots$ is uniquely determinded. Hence $\tilde{g}_0(c_kz^k)$ is a solution of (1.2) with $\tilde{g}_0(c_kz^k) \equiv c_kz^k \pmod{z^{k+1}}$.
It remains to show that for every $c_k \in \mathbb{C}^\star$ there exists at most one solution $g(z) \equiv c_kz^k \pmod{z^{k+1}}$ mit $g(z) = \tilde{g}_0(c_kz^k)$. Let $g_j(z) = c_kz^k + \ldots + c_{l-1}z^{l-1} + c_l^{(j)}z^l + \ldots$, for $j = 1,2$ with $l > k$ be solutions of (1.2) with $c_l^{(1)} \neq c_l^{(2)}$. Both functions, namely

g_1 and g_2 fulfill the transformed generalized Dhombres functional equation (1.2) $g(w_0z + zg(z)) = \tilde{\varphi}(g(z))$ and therefore we get

$$\begin{aligned} g_j(zw_0 + zg_j(z)) = c_k(zw_0 + zg_j(z)) + \ldots + c_{l-1}(zw_0 + zg_j(z))^{l-1} \\ + c_l^{(j)}(zw_0 + zg_j(z))^l + \ldots \end{aligned}$$

for $j = 1, 2$. In every bracket the term $zg_j(z)$ appears and hence the first term of $g_1(z) - g_2(z)$ which is different from zero is $w_0^l(c_l^{(1)} - c_l^{(2)})z^l$. The right hand side of (1.2) is given by

$$\tilde{\varphi}(g_j(z)) = w_0^k g_j(z) + \ldots = w_0^k c_k z^k + \ldots + w_0^k c_l^{(1)} z^l + \ldots$$

and so we have

$$\tilde{\varphi}(g_1(z)) - \tilde{\varphi}(g_2(z)) = w_0^k(c_l^{(1)} - c_l^{(2)})z^l + \ldots .$$

Now we compare the coefficients of z^l and we use that $c_l^{(1)} - c_l^{(2)} \neq 0$. This leads to $w_0^k = w_0^l$ and so $w_0^{l-k} = 1$ but this is a contradiction because the complex number w_0 is no root of unity and hence there exist at most one solution g for $c_k \in \mathbb{C}^\star$. $\square$

3. Infinite products

In this section we investigate solutions of the generalized Dhombres functional equation which are represented as infinite products. From Theorem 1 in [4] on page 828 it is known that if $w_0 \neq 0$ and if the function $\tilde{\varphi}$ is local analytic in a sufficiently small neighbourhood of zero, then also the function $\tilde{g}_0$ and hence the function f are local analytic in a sufficiently small neighbourhood of zero, respectively w_0. We will use this in the proof of the following theorem.

Theorem 3.1. *Let the function $\tilde{\varphi}$ be given by $\tilde{\varphi}(y) = w_0^k y + \ldots$ and let $\tilde{\varphi}$ be local analytic in $|y| < r$ with $r > 0$, and let the complex number w_0 fulfill $0 < |w_0| < 1$. Then the function $\tilde{g}_0$ has the representation*

$$\tilde{g}_0(y) = \left[S_{\tilde{\varphi}}^{-1}(y) \prod_{\nu=0}^{\infty} \frac{1}{\left(1 + w_0^{-1}\tilde{\varphi}^\nu(y)\right)^k} \right]^{[-1]}$$

where $^{[-1]}$ denotes the inverse with respect to substitution.

Proof. We iterate the equation

$$F(z) + G(z) = G(w_0^k z) \tag{2.8}$$

which we obtained in the proof of the previous theorem. Firstly we assume that (2.8) has a solution G which is analytic in $|y| < r$. In a first step we substitute $w_0^k z$ for z and therefore we get

$$F(w_0^k z) + F(z) + G(z) = G(w_0^{2k} z).$$

It is easy to show that by induction we get

$$\sum_{\nu=0}^{n-1} F(w_0^{\nu k} y) + G(y) = G(w_0^{nk} y).$$

We know from the previous proof that $G(y)$ is local analyic in $|y| < r$ because $\tilde{\varphi}(y)$ is local analytic in a sufficiently small neighbourhood of zero and $G(0) = 0$. Well known elementary estimates show that

$$\lim_{n\to\infty} |G(w_0^{nk} y)| = 0$$

in every compact subset of $|y| < r$, this convergence is uniform, and hence we obtain

$$G(y) = -\sum_{\nu=0}^{\infty} F(w_0^{\nu k} y). \tag{3.1}$$

Now we show that (3.1) gives indeed a local analytic solution of (2.8). Using the definition of F and similar estimates as above we see that the series on the right hand side $-\sum_{\nu=0}^{\infty} F(w_0^{\nu k} y)$ of (3.1) is uniformly convergent in every compact subset of $|y| < r$ and also absolutely convergent. Hence, by Weierstrass' theorem G is analytic in $|y| < r$, and it is easy to check that it is a solution of (2.8). Previously we used $Q^*(y) = \exp(G(y))$ and hence we obtain

$$Q^*(y) = \exp(G(y)) = \exp(-\sum_{\nu=0}^{\infty} F(w_0^{\nu k} y)) = \prod_{\nu=0}^{\infty} \frac{1}{(1 + w_0^{-1} S_{\tilde{\varphi}}(w_0^{\nu k} y))^k}$$

and therefore

$$Q(y) = y \prod_{\nu=0}^{\infty} \frac{1}{(1 + w_0^{-1} S_{\tilde{\varphi}}(w_0^{\nu k} y))^k}.$$

This expression is absolutely and uniformly convergent in every compact subset of $|y| < r$ which follows from a more detailed discussion of $\sum_{\nu=0}^{\infty} kLn(1 + w_0^{-1} S_{\tilde{\varphi}}(w_0^{\nu k} y))$. We can use the series representation of the formal loagrithm and so it is possible to represent the term $kLn(1 + w_0^{-1} S_{\tilde{\varphi}}(w_0^{\nu k} y))$ as

$$kLn(1 + w_0^{-1} S_{\tilde{\varphi}}(w_0^{\nu k} y)) = w_0^{\nu k} t_1 y + w_0^{2\nu k} t_2 y^2 + w_0^{3\nu k} t_3 y^3 + \ldots$$

where the t_j with $j \geq 1$ are formed by w_0 and by coefficients of $S_{\tilde{\varphi}}$. So the statement follows because of the term $w_0^{l\nu k}$, $l \geq 1$. In the previous proof we used $Q \circ S_{\tilde{\varphi}}^{-1} = \tilde{h}_0$ and therefore we obtain

$$\tilde{h}_0(y) = Q(S_{\tilde{\varphi}}^{-1}(y)) = S_{\tilde{\varphi}}^{-1}(y) \prod_{\nu=0}^{\infty} \frac{1}{(1 + w_0^{-1} S_{\tilde{\varphi}}(w_0^{\nu k} S_{\tilde{\varphi}}^{-1}(y)))^k)}$$

where this expression is absolutely and uniformly convergent in every compact subset of $|y| < r$ which we want to prove now. The product is convergent, therefore let $r > 0$ and $\tilde{\varphi}(y)$ local analytic in $|y| < r$. Then according to Theorem 2.1 the function $S_{\tilde{\varphi}}$ is local analyic in a sufficiently small neighbourhood of zero and hence also $S_{\tilde{\varphi}}^{-1}$ is. Therefore we obtain the absolute and uniform convergence. Next we have to show that there exist a $\tilde{r} > 0$ such that $\tilde{\varphi}^{\nu}$ is defined for a $\nu \geq 1$ and such that $\tilde{\varphi}^{\nu}$ is local analytic in $|y| < \tilde{r}$. Furthermore the function $\tilde{\varphi}^{\nu}(y)$ has to fulfill $\tilde{\varphi}^{\nu}(y) = S_{\tilde{\varphi}}(w_0^{\nu k} S_{\tilde{\varphi}}^{-1}(y))$ in $|y| < \tilde{r}$. Then as a consequence of $\tilde{h}_0 = \tilde{g}_0^{-1}$ the assertion

$$\tilde{g}_0(y) = \left[S_{\tilde{\varphi}}^{-1}(y) \prod_{\nu=0}^{\infty} \frac{1}{\left(1 + w_0^{-1} \tilde{\varphi}^{\nu}(y)\right)^k} \right]^{[-1]},$$

where the convergence of the right hand side is absolute and uniform in a sufficiently small neighbourhood of zero, follows. Let $\tilde{\varphi}(y)$ be local analytic in $|y| < r$, then every iterate $\tilde{\varphi}^{\nu}(y)$, $\nu \in \mathbb{N}$ starts with the term $w_0^{\nu k} y$. Hence with the same arguments we used before we get that every iterate $\tilde{\varphi}^{\nu}(y)$, $\nu \in \mathbb{N}$, is local analytic in $|y| < r$. It remains to show that $\tilde{\varphi}^{\nu}(y) = S_{\tilde{\varphi}}(w_0^{\nu k} S_{\tilde{\varphi}}^{-1}(y))$ is valid. Using induction we know that for $\nu = 1$ the assertion is exactly the assertion of Theorem 2.1. Then we obtain

$$\begin{aligned}\tilde{\varphi}^{\nu+1}(y) &= \tilde{\varphi}^{\nu}(\tilde{\varphi}(y)) = S_{\tilde{\varphi}}(w_0^{\nu k} S_{\tilde{\varphi}}^{-1}(\tilde{\varphi}(y))) = S_{\tilde{\varphi}}(w_0^{\nu k} S_{\tilde{\varphi}}^{-1}(S_{\tilde{\varphi}}(w_0^{k} S_{\tilde{\varphi}}^{-1}(y)))) \\ &= S_{\tilde{\varphi}}(w_0^{\nu k} w_0^{k} S_{\tilde{\varphi}}^{-1}(y)) = S_{\tilde{\varphi}}(w_0^{(\nu+1) k} S_{\tilde{\varphi}}^{-1}(y))\end{aligned}$$

and hence the theorem is proved. □

After this representation of the solution as infinite product in the sense of complex analysis we want to present another representation where we use the weak topology in the ring of formal series. We emphasize that these product representations are valid for formal solutions, and also the given $\tilde{\varphi}$ is understood as formal series. For an introduction to the weak topology we refer the reader to [6] chapter 22. Before we formulate the theorem we will proof a lemma. By $w - \lim$ we denote the limit according to the weak topology. We only recall the following fact. Let $(F_n(y))_{n \geq 1}$ be a sequence of formal series, $F_n(y) = \sum_{\nu=0}^{\infty} c_{\nu}^{(n)} y^{\nu}$ and $F(y) = \sum_{\nu=0}^{\infty} c_{\nu} y^{\nu}$, then $w - \lim_{n\to\infty} F^{(n)} = F$ if and only if $\lim_{n\to\infty} c_{\nu}^{(n)} = c_{\nu}$, for all $\nu \geq 0$.

Lemma 3.2. *Let $(F_n)_{n\geq 1}$ and $(G_n)_{n \geq 1}$ be sequences in $\mathbb{C}[\![y]\!]$ such that ord $F_n \geq 1$ and ord $G_n = 0$ for all $n \in \mathbb{N}$ and such that $w - \lim_{n\to\infty} F_n = F$ as well as $w - \lim_{n\to\infty} G_n = G$ exist such that ord $G \geq 0$. The function G_n^{-1} denotes the inverse of G_n with respect to multiplication. Furthermore let $\Phi_n \in \mathbb{C}[\![y]\!]$ and let $w - \lim_{n\to\infty} \Phi_n = \Phi$ exist. Then*

1. $w - \lim_{n\to\infty} \Phi_n(F_n(y)) = \Phi(F(y)) = \Phi(w - \lim_{n\to\infty} F_n(y))$,
2. $w - \lim_{n\to\infty} G_n^{-1}(y) = G^{-1}$.

Proof. (1) The series Φ_n and F_n are given as $\Phi_n(w) = \varphi_0^{(n)} + \varphi_1^{(n)} w + \varphi_2^{(n)} w^2 + \ldots$ and $F_n(y) = c_1^{(n)} y + c_2^{(n)} y^2 + \ldots$. Then we obtain

$$\begin{aligned}F_n(y)^l &= \left(c_1^{(n)}\right)^l y^l + l \left(c_1^{(n)}\right)^{l-1} c_2^{(n)} y^{l+1} + \ldots \\ &= \left(c_1^{(n)}\right)^l y^l + \ldots + R_{l,\mu}\left(c_1^{(n)}, \ldots, c_{\mu-l+1}^{(n)}\right) y^{\mu} + \ldots,\end{aligned}$$

where $R_{l,\mu}$, for $\mu \geq 2$ is a polynomial. The expression

$$\begin{aligned}\Phi_n\left(F_n(y)\right) &= \varphi_0^{(n)} + \varphi_1^{(n)} F_n(y) + \varphi_2^{(n)} F_n(y)^2 + \ldots \\ &= \varphi_0^{(n)} + \sum_{l \geq 1} \varphi_l^{(n)} \left(\sum_{\mu \geq l} R_{l,\mu}\left(c_1^{(n)}, \ldots, c_{\mu-l+1}^{(n)}\right) y^{\mu}\right) \\ &= \varphi_0^{(n)} + \sum_{\mu=1}^{\infty} \left(\sum_{l \leq \mu} \varphi_l^{(n)} R_{l,\mu}\left(c_1^{(n)}, \ldots, c_{\mu-l+1}^{(n)}\right)\right) y^{\mu}\end{aligned}$$

follows and hence we get

$$\lim_{n\to\infty} \sum_{l \leq \mu} \varphi_l^{(n)} R_{l,\mu}\left(c_1^{(n)}, \ldots, c_{\mu-l+1}^{(n)}\right) = \sum_{l \leq \mu} \varphi_l R_{l,\mu}\left(c_1, \ldots, c_{\mu-l+1}\right).$$

This means that the limit $w-\lim_{n\to\infty}\Phi_n(F_n(y))$ exists and that

$$w-\lim_{n\to\infty}\Phi_n(F_n(y))=\Phi(F(y))$$

is true.
(2) We represent G_n and G_n^{-1} as $G_n(y)=d_0^{(n)}+d_1^{(n)}y+\dots$ where $d_0^{(n)}\neq 0$ and $G_n^{-1}(y)=h_0^{(n)}+h_1^{(n)}y+\dots$. The relation $G_n(y)G_n^{-1}(y)=1$ is fulfilled and hence $d_0^{(n)}h_0^{(n)}=1$ which is the same as

$$h_0^{(n)}=\frac{1}{d_0^{(n)}}.$$

The limit of $d_0^{(n)}$ for n tending to ∞ exists and is equal to $d_0\neq 0$ therefore also h_0 exists and we obtain

$$h_0=\lim_{n\to\infty}h_0^{(n)}=\lim_{n\to\infty}\frac{1}{d_0^{(n)}}=\frac{1}{d_0}.$$

For $j>1$ we have

$$h_j^{(n)}=\frac{1}{d_0^{(n)}}P_j(c_1^{(n)},\dots,c_j^{(n)},h_0^{(n)},\dots,h_{j-1}^{(n)}),$$

where P_j denotes a polynomial. Therefore h_j exists and has the form

$$h_j=\frac{1}{d_0}P_j(c_1,\dots,c_j,h_0,\dots,h_{j-1}).$$

Finally

$$w-\lim_{n\to\infty}G_n^{-1}(y)=G^{-1}(y)$$

follows. □

Now we want to prove a theorem concerning a product representation in the weak topology. Note that in the following theorem it is not required that the function $\tilde{\varphi}$ is local analytic.

Theorem 3.3. *Let $0<|w_0|<1$, then $\tilde{g}_0$ has a representation of the form*

$$\tilde{g}_0(y)=\left[S_{\tilde{\varphi}}^{-1}(y)\prod_{\nu=0}^{\infty}\frac{1}{\left(1+w_0^{-1}\tilde{\varphi}^{\nu}(y)\right)^k}\right]^{[-1]}$$

where $^{[-1]}$ denotes the inverse with respect to substitution and

$$\prod_{\nu=0}^{\infty}\frac{1}{\left(1+w_0^{-1}\tilde{\varphi}^{\nu}(y)\right)^k}=w-\lim_{n\to\infty}\prod_{\nu=0}^{n}\frac{1}{\left(1+w_0^{-1}\tilde{\varphi}^{\nu}(y)\right)^k}$$

Proof. To prove this theorem we iterate the equation

$$(1+w_0^{-1}S_{\tilde{\varphi}}(y))^kQ^*(y)=Q^*(w_0^ky). \tag{2.7}$$

This equation we obtained in the second proof of Theorem 1.1. Remember that $S_{\tilde{\varphi}}$ is according to Theorem 2.1 the unique solution of the linearization equation and the function $Q^*(y)$ is constructed such that $Q^*(y)=1+\tilde{q}_1y+\tilde{q}_2y^2+\dots$. In a first iteration step we substitute w_0^ky for y and so we get

$$(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^ky))^k(1+w_0^{-1}S_{\tilde{\varphi}}(y))^kQ^*(y)=Q^*(w_0^{2k}y).$$

Inductively we obtain

$$\prod_{\nu=0}^{n-1}(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k Q^*(y) = Q^*(w_0^{nk}y).$$

Next we want to show that $Q^*(w_0^{nk}y)$ converges to 1 in the weak topology, if n tends to infinity. Using the representation of Q^* leads to

$$Q^*(w_0^{nk}y) = 1 + w_0^{nk}\tilde{q}_1 y + w_0^{2nk}\tilde{q}_2 y^2 + \dots.$$

Then for $\nu \geq 1$ we obtain

$$\lim_{n\to\infty} \tilde{q}_\nu w_0^{\nu nk} = 0,$$

and with the notation $w-\lim$ for the limit with respect to the weak topology we get

$$w - \lim_{n\to\infty} Q^*(w_0^{nk}y) = 1.$$

Therefore the product $\prod_{\nu=0}^{\infty}(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k$ exists in the weak topology and

$$Q^*(y) = \prod_{\nu=0}^{\infty} \frac{1}{(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k}.$$

Hence the solution $Q^*(y)$ of the functional equation

$$(1+w_0^{-1}S_{\tilde{\varphi}}(y))^k Q^*(y) = Q^*(w_0^k y) \tag{2.7}$$

is uniquely given by this product, provided a solution of (2.7) exists.
In a next step we want to show that the product

$$\prod_{\nu=0}^{\infty}(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k$$

converges with respect to the weak topology and hence that

$$Q^*(y) = \prod_{\nu=0}^{\infty} \frac{1}{(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k}$$

is a solution of the functional equation (2.7). According to Lemma 3.2 it is sufficient to prove that $\sum_{\nu=0}^{\infty} Ln(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k$ exists in the weak topology. We discuss $kLn(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))$ and so, if we use the series representation of the logarithm, we obtain

$$kLn(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y)) = k\sum_{l=1}^{\infty}(-1)^{l-1}\frac{(w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^l}{l}.$$

Computing this expression leads to

$$\begin{aligned} kLn(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y)) &= w_0^{\nu k}(kw_0^{-1})y + w_0^{2\nu k}(+ks_2w_0^{-1} - k\frac{w_0^{-2}}{2})y^2 \\ &\quad + w_0^{3\nu k}(+ks_3w_0^{-1} - 2ks_2\frac{w_0^{-2}}{2} + k\frac{w_0^{-3}}{3})y^3 + \dots. \end{aligned}$$

Now we consider the representation $\sum_{\nu=0}^{n}\sum_{l=1}^{\infty}(w_0^{\nu k})^l p_l y^l$, where the first summation comes from the infinite product and the second is introduced by the logarithm. Then the equation

$$\sum_{\nu=0}^{n}\sum_{l=1}^{\infty}(w_0^{\nu k})^l p_l y^l = \sum_{l=1}^{\infty}\left(\sum_{\nu=0}^{n}(w_0^{\nu k})^l p_l\right) y^l$$

holds. The p_l's are given by $p_1 = kw_0^{-1}, p_2 = ks_2w_0^{-1} - k\frac{w_0^{-2}}{2}, p_3 = ks_3w_0^{-1} - 2ks_2\frac{w_0^{-2}}{2} + k\frac{w_0^{-3}}{3}, \ldots, p_l = ks_lw_0^{-1} - \ldots + k\frac{w_0^{-l}}{l}$ for $l \geq 1$ and hence we have

$$w_0^{\nu kl}p_l = +ks_l(w_0^{\nu k})^{l-1} - \ldots + \frac{k}{l}w_0^{\nu k} = ks_l(w_0^{(l-1)k})^\nu - \ldots + \frac{k}{l}(w_0^k)^\nu.$$

Now we obtain

$$\lim_{n\to\infty}\sum_{\nu=0}^{n}(w_0^{\nu k})^l p_l = ks_l\sum_{\nu=0}^{\infty}(w_0^{(l-1)k})^\nu - \ldots + \frac{k}{l}\sum_{\nu=0}^{\infty}(w_0^k)^\nu$$

where the expressions on the right hand side of the equation are by the assumption $|w_0| < 1$ convergent geometric series, and hence the product

$$\prod_{\nu=0}^{\infty}\frac{1}{(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k}$$

exists with respect to the weak topology and $Q^*(y) = \prod_{\nu=0}^{\infty}\frac{1}{(1+w_0^{-1}S_{\tilde{\varphi}}(w_0^{\nu k}y))^k}$ is a solution of the functional equation (2.7). Finally we reverse the transformations which we did, namely $Q(y) = yQ^*(y)$ and $\tilde{g}_0^{-1}(y) = Q(S_{\tilde{\varphi}}^{-1}(y))$ and note that also the $\nu - th$ iterate of $\tilde{\varphi}$ is again $S_{\tilde{\varphi}}(w_0^{\nu k}S_{\tilde{\varphi}}^{-1}(y))$. Hence for $\tilde{g}_0$ we get

$$\tilde{g}_0(y) = \left[S_{\tilde{\varphi}}^{-1}(y)\prod_{\nu=0}^{\infty}\frac{1}{\left(1+w_0^{-1}\tilde{\varphi}^\nu(y)\right)^k}\right]^{[-1]},$$

where the product on the right hand side is convergent with respect to the weak topology. □

Remark 3.4. The solutions f of the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

where $f(0) = w_0$ and $0 < |w_0| < 1$ are given by

$$f(z) = w_0 + \left[S_{\tilde{\varphi}}^{-1}(y)\prod_{\nu=0}^{\infty}\frac{1}{\left(1+w_0^{-1}\tilde{\varphi}^\nu(y)\right)^k}\right]^{[-1]}$$

where we also have to take into account the different assumptions which we used in Theorem 3.1 respectively in Theorem 3.3.

So we constructed the solutions f of the generalized Dhombres functional equation where $f(0) = w_0$ and $|w_0| < 1$. Above all the weak topology is not often used and so we can ask whether there is another possibility to apply this topology in product representations for solutions f where $f(z_0) = w_0$ and $z_0 \neq 0$?
For these considerations let us start with some transformations. We consider the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

and we are looking for solutions f of (1.1) with $f(z_0) = w_0$. Then the function f can be represented as $f(z) = w_0 + \tilde{g}(z)$ where the function $\tilde{g}$ fulfills $\tilde{g}(z_0) = 0$ and $\tilde{g}(z) = \tilde{g}(z_0+\zeta) =: g(\zeta)$ with $g(\zeta) = c_k\zeta^k + \ldots$, $k \geq 1$ and ζ is defined as $\zeta = z - z_0$. Constant solutions are not of importance for us and therefore there exists a natural number k such that $c_k \neq 0$. The function φ can be written as $\varphi(w) = \varphi(w_0) + \tilde{\varphi}(\omega)$

with $\omega = w - w_0$ and $\tilde{\varphi}(w_0) = 0$ and $\tilde{\varphi}(y) = d_1 y + d_2 y^2 + \ldots$. These presentments are substituted in (1.1) and hence after some transformations we obtain

$$g((z_0 w_0 - z_0) + z_0 g(\zeta) + \zeta w_0 + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)). \tag{3.2}$$

This equation can be treated by our formal power series methods if $z_0 w_0 - z_0 = 0$. We will investigate the case where $w_0 = 1$ and $z_0 \neq 0$. That means we are looking for solutions f of (1.1) with $f(z_0) = 1$ and $z_0 \neq 0$. So, after using $z_0 w_0 - z_0 = 0$ equation (3.2) is the same as

$$g(z_0 g(\zeta) + \zeta w_0 + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)) \tag{3.3}$$

and furthermore with $T(\zeta)^k = g(\zeta)$ respectively $\hat{\varphi}(z)^k = \tilde{\varphi}(z^k)$ and $T^{-1} = U$ we obtain

$$z_0 \zeta^k + (w_0 + \zeta^k) U(\zeta) = U(\hat{\varphi}(\zeta)).$$

The situation where $z_0 \neq 0$ and $w_0 = 1$ leads to

$$z_0 \zeta^k + (1 + \zeta^k) U(\zeta) = U(\tilde{\varphi}(\zeta)). \tag{3.4}$$

Comparing coefficients of equation (3.4) indicates that the natural number k has to be one or else we will not get a solution. Therefore we can consider

$$z_0 \zeta + (1 + \zeta) U(\zeta) = U(\tilde{\varphi}(\zeta)) \tag{3.5}$$

where $\tilde{\varphi}(\zeta) = \hat{\varphi}(\zeta)$ and for further calculations we will assume that $\tilde{\varphi}(\zeta) = d_1 \zeta + \ldots$ where the coefficient d_1 is different from zero. We need the following transformation which we fomulate as remark

Remark 3.5. Using the transformation $U(\zeta) = \tilde{V}(\zeta) - z_0$ in (3.5) leads to the linear functional equation

$$(1 + \zeta) V(\zeta) = V(\tilde{\varphi}(\zeta)) \tag{3.6}$$

where $\tilde{V}(\zeta) = z_0 V(\zeta)$.

Proof. Substituting $\tilde{V}(\zeta) - z_0$ for $U(\zeta)$ in (3.5) leads to

$$z_0 \zeta + \tilde{V}(\zeta) + \zeta \tilde{V}(\zeta) - z_0 - \zeta z_0 = \tilde{V}(\tilde{\varphi}(\zeta)) - z_0$$

and this is the same as

$$(1 + \zeta) \tilde{V}(\zeta) = \tilde{V}(\tilde{\varphi}(\zeta)).$$

Writing $\tilde{V}(\zeta) = z_0 V(\zeta)$ where $V(\zeta) = 1 + \ldots$ results in

$$(1 + \zeta) V(\zeta) = V(\tilde{\varphi}(\zeta)).$$

□

We get the following theorem.

Theorem 3.6. *Let $\tilde{\varphi}(\zeta) = d_1 \zeta + \ldots$ be local analytic in a sufficiently small neighbourhood of zero and $0 < |d_1| < 1$. Then the function g represented as*

$$g(\zeta) = \left[z_0 \prod_{\nu=0}^{\infty} \frac{1}{(1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}}(\zeta)))} - z_0 \right]^{[-1]}$$

where $^{[-1]}$ denotes the inverse with respect to substitution is a local analytic solution of the functional equation

$$g(z_0 g(\zeta) + \zeta w_0 + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)). \tag{3.3}$$

Sketch of the proof. The absolute value of the first coefficient d_1 of the local analytic function $\tilde{\varphi}$ is smaller than one and hence according to Theorem 2.1 there exists a local analytic function $S_{\tilde{\varphi}}$ such that $\tilde{\varphi}(\zeta) = S_{\tilde{\varphi}}(d_1 S_{\tilde{\varphi}}^{-1}(\zeta))$. We substitute this representation of $\tilde{\varphi}$ in the linear functional equation

$$(1+\zeta)V(\zeta) = V(\tilde{\varphi}(\zeta)) \tag{3.6}$$

and so we obtain

$$(1+\zeta)V(\zeta) = V(S_{\tilde{\varphi}}(d_1 S_{\tilde{\varphi}}^{-1}(\zeta))).$$

Next we substitute $S_{\tilde{\varphi}}(\zeta)$ for ζ and by W we denote the function $V \circ S_{\tilde{\varphi}}$. Hence we get

$$(1+S_{\tilde{\varphi}}(\zeta))W(\zeta) = W(d_1\zeta)$$

where $W(\zeta) = 1 + \ldots$. We define the functions F and G by $F(\zeta) := Ln(1+S_{\tilde{\varphi}}(\zeta))$ and $G(\zeta) := LnW(\zeta)$. Then the equivalent equation

$$F(\zeta) + G(\zeta) = G(d_1\zeta)$$

remains. An iteration process which is done by induction leads to

$$G(\zeta) = -\sum_{\nu=0}^{\infty} F(d_1^{\nu}\zeta)$$

where the absolute and uniform convergence follows, like in the proof of Theorem 3.1, from the fact that $0 < |d_1| < 1$. So we can reverse our transformations and then we obtain

$$W(\zeta) = exp\, G(\zeta) = \prod_{\nu=0}^{\infty} \frac{1}{expF(d_1^{\nu}\zeta)} = \prod_{\nu=0}^{\infty} \frac{1}{1+S_{\tilde{\varphi}}(d_1^{\nu}\zeta)}$$

and because of $V(\zeta) = W(S_{\tilde{\varphi}}^{-1}(\zeta))$ and $\tilde{V} = z_0 V$ we get

$$\tilde{V}(\zeta) = z_0 \prod_{\nu=0}^{\infty} \frac{1}{1+S_{\tilde{\varphi}}(d_1^{\nu} S_{\tilde{\varphi}}^{-1}(\zeta))}.$$

For U we get

$$U(\zeta) = z_0 \prod_{\nu=0}^{\infty} \frac{1}{1+S_{\tilde{\varphi}}(d_1^{\nu} S_{\tilde{\varphi}}^{-1}(\zeta))} - z_0$$

and hence

$$g(\zeta) = \left[z_0 \prod_{\nu=0}^{\infty} \frac{1}{1+S_{\tilde{\varphi}}(d_1^{\nu} S_{\tilde{\varphi}}^{-1}(\zeta))} - z_0 \right]^{[-1]}.$$

□

In the next theorem we consider the situation of the convergence of the product with respect to the weak topology. Note that $\tilde{\varphi}$ is not necessarily convergent.

Theorem 3.7. *Let $\tilde{\varphi}(\zeta) = d_1\zeta + \ldots$ such that $0 < |d_1| < 1$. Then the series g represented as*

$$g(\zeta) = \left[z_0 \prod_{\nu=0}^{\infty} \frac{1}{\left(1+S_{\tilde{\varphi}}(d_1^{\nu} S_{\tilde{\varphi}}^{-1}(\zeta))\right)} - z_0 \right]^{[-1]}$$

where $^{[-1]}$ *denotes the inverse with respect to substitution is a solution of the functional equation*

$$g(z_0 g(\zeta) + \zeta w_0 + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)). \tag{3.3}$$

where the infinite product is understood in the weak topology,

$$\prod_{\nu=0}^{\infty} \frac{1}{\left(1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}}^{-1}(\zeta))\right)} = w - \lim_{n\to\infty} \prod_{\nu=0}^{n} \frac{1}{\left(1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}}^{-1}(\zeta))\right)}.$$

Sketch of the proof. Like in the previous proof we can transform the equation

$$(1+\zeta)V(\zeta) = V(\tilde{\varphi}(\zeta)) \tag{3.6}$$

to $(1 + S_{\tilde{\varphi}}(\zeta))W(\zeta) = W(d_1\zeta)$ where W is defined as $W(\zeta) := V(S_{\tilde{\varphi}}(\zeta))$. An iteration process leads to

$$\prod_{\nu=0}^{n-1}(1 + S_{\tilde{\varphi}}(d_1^\nu \zeta))W(\zeta) = W(d_1^n \zeta).$$

Then we have $W(d_1^n\zeta) = 1 + \lambda_1 d_1^n \zeta + \lambda_2 d_1^{n^2}\zeta^2 + \dots$ and so the function $W(d_1^n\zeta)$ converges to one in the weak topology and we obtain necessarily

$$\tilde{V}(\zeta) = z_0 \prod_{\nu=0}^{\infty} \frac{1}{1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}^{-1}}(\zeta))}$$

if (3.6) has a solution. In order to show that there is indeed a solution, we can again consider the logarithmic series $\sum_{\nu=0}^{\infty} Ln(1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}}^{-1}(\zeta)))$ and so we get

$$Ln(1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}}^{-1}(\zeta))) = \tilde{s}_1 d_1^\nu \zeta + \tilde{s}_2 d_1^{2\nu}\zeta^2 + \tilde{s}_3 d_1^{3\nu}\zeta^3 + \dots$$

where we unite the coefficients of the logarithmic series and of $S_{\tilde{\varphi}}$ respectively $S_{\tilde{\varphi}}^{-1}$ under the coefficients $\tilde{s}_\nu$, $\nu \geq 1$. We consider $\sum_{l=1}^{\infty} \tilde{s}_l \left(\sum_{\nu=0}^{n} d_1^{\nu l}\right)\zeta^l$ and so we get

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} (d_1^l)^\nu = \sum_{\nu=0}^{\infty} (d_1^l)^\nu$$

and this geometric series converges since $|d_1^l| < 1$. Hence the product converges with respect to the weak topology. Again we obtain

$$g(\zeta) = \left[z_0 \prod_{\nu=0}^{\infty} \frac{1}{\left(1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}}^{-1}(\zeta))\right)} - z_0 \right]^{[-1]}$$

but now this is convergent in the weak topology. □

Remark 3.8. With respect to the different assumptions of Theorem 3.6 and Theorem 3.7 we obtain local analytic solutions f or solutions f which are convergent with respect to the weak topology of the generalized Dhombres functional equation with the representation

$$f(z) = w_0 + \left[z_0 \prod_{\nu=0}^{\infty} \frac{1}{(1 + S_{\tilde{\varphi}}(d_1^\nu S_{\tilde{\varphi}}(\zeta)))} - z_0 \right]^{[-1]}$$

where $\zeta = z - z_0$ and $z_0 \neq 0$.

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