FORMAL SOLUTIONS OF THE GENERALIZED DHOMBRES FUNCTIONAL EQUATION WITH VALUE ONE AT ZERO

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Abstract. We study formal solutions f of the generalized Dhombres functional equation $f(zf(z)) = \varphi(f(z))$. In contrary to the situation where $f(0) = w_0$ and $w_0 \in \mathbb{C} \setminus \mathbb{E}$ where $\mathbb{E}$ denotes the complex roots of 1, which were already discussed, we investigate solutions f where $f(0) = 1$. To obtain solutions in this case we use new methods which differ from the already existing ones.

1. Introduction

We study the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

in the complex domain. The function φ is known and we are looking for local analytic or formal solutions f of (1.1). The original Dhombres functional equation in the real domain is given by

$$f(zf(z)) = f(z)^2.$$

This equation was first discussed by Jean Dhombres, see [2]. In the complex domain equation (1.1) was introduced by L. Reich, J. Smítal and M. Štefánková in [4]. In [4] they were looking for solutions f of (1.1) with $f(0) = 0$ and in the subsequent paper [5] the authors discussed solutions f with $f(0) = w_0$ where w_0 is a complex number different from zero and also no complex root of unity. If $f(0) = 1$ or more generally $f(0) = w_0$ and $w_0 \neq 0$ but f not constant, we can write $f(z) = w_0 + g(z)$ where $g(z) = c_k z^k + c_{k+1} z^{k+1} + \dots$ and $k \in \mathbb{N}$, $c_k \neq 0$. Substituting this representation of f in (1.1) leads to the transformed generalized Dhombres functional equation

$$g(w_0 z + zg(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

where the function $\tilde{\varphi}$ is computed by $\varphi(y) = w_0 + \tilde{\varphi}(y - w_0)$ and hence it is given by $\tilde{\varphi}(y) = w_0^k y + d_2 y^2 + \dots$ which can be shown by comparing the coefficients of (1.2).

Let us introduce some notations we will use, an introduction to formal power series can be found in the book [1] of H. Cartan.

Definition 1.1. By

$$\mathbb{C}[\![z]\!] = \left\{ F | F(z) = a_0 + a_1 z + a_2 z^2 + \dots,\ a_\nu \in \mathbb{C},\ \nu \geq 0 \right\}$$

we denote the set of formal power series. For a series $F \in \mathbb{C}[\![z]\!]$ the order of F is defined by $ord\, F := min\{\nu \in \mathbb{N} | a_\nu \neq 0\}$ and one sets ∞ for the order of the trivial

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series. We have

$$\mathbb{C}[\![z^k]\!] = \{F|F \in \mathbb{C}[\![z]\!], F(z) = \sum_{\nu \geq 0} a_\nu z^\nu, a_\nu = 0 \text{ if } \nu \not\equiv 0 \pmod{k}\}$$

for $k \in \mathbb{N}$. A series $F \in z\mathbb{C}[\![z^k]\!]$ has the representation $F(z) = a_1 z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \ldots$. We also need

$$\Gamma = \{F|F \in \mathbb{C}[\![z]\!] \text{ and } ord\ F = 1\}$$

and $(\Gamma, \circ)$ is the group of invertible formal power series with respect to substitution $\circ$ in $\mathbb{C}[\![z]\!]$. Then the set

$$\Gamma_1 = \left\{F|F \in \Gamma \text{ and } F(z) \equiv z \pmod{z^2}\right\}$$

is a subgroup of $(\Gamma, \circ)$. Finally we define

$$\mathbb{C}^{(k)}[\![z]\!] = \{F\big|F \in \mathbb{C}[\![z]\!], F(z) = \sum_{\nu \geq 0} c_\nu z^\nu \text{ where } c_\nu = 0 \text{ if } \nu \not\equiv 1 \pmod{k}\}$$

which leads to $\Gamma^{(k)} = \Gamma \cap \mathbb{C}^{(k)}[\![z]\!]$ and $\Gamma_1^{(k)} = \Gamma_1 \cap \mathbb{C}^{(k)}[\![z]\!]$, we will also use $\mathbb{C}^{(k)}[\![z]\!] = z\mathbb{C}[\![z^k]\!]$. We set $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$.

With this definition we can formulate the theorem which we want to prove. From now on we are mainly interested in formal solutions f of (1.1) with $f(0) = 1$, or equivalently, with formal solutions g of (1.2) which are not 0. Concerning local analytic solutions we can only present one example at the end of the paper. It is an open question whether the methods for constructing local analytic solutions of certain non linear functional equations presented in [7] and [8] can be also used for the convergence problems related to generalized Dhombres functional equations. It should be mentioned that the results presented in Theorem 1.2 were partially used already in [6] (Lemma 4), which is devoted to non constant polynomial solutions of (1.1). However, in [6] Lemma 4, the existence of a non constant solution is supposed, whereas our Theorem 1.2 gives a **necessary and sufficient condition** on $\tilde{\varphi}$ for the existence of non constant formal solutions of (1.1) with $f(0) = 1$, namely that

$$\tilde{\varphi}(y) = y + ky^2 + \ldots,\ k \in \mathbb{N},$$

holds. If $\tilde{\varphi}$ has this form, then in the proof of Theorem 1.2 formal solutions f with $f(0) = 1$ are constructed. We mention here, without proof, that in general the necessary and sufficient conditions on $\tilde{\varphi}$ for the existence of formal solutions f of (1.1) with $f(0) \in \mathbb{E}$ are much more complicate. Hence the authors believe that the case $f(0) = 1$ deserves a seperate treatment. Our proof of Theorem 1.2 is in several places different from the rather sketching one of Lemma 4 in [6].

Theorem 1.2. *Let $w_0 = 1$ and let the formal series $\tilde{\varphi}$ be given by $\tilde{\varphi}(y) = y + ky^2 + \ldots$ for a $k \in \mathbb{N}$. Then there exists a unique formal series $\tilde{g}_0 \in \Gamma_1$ such that the set of non constant soutions g of the transformed generalized Dhombres functional equation*

$$g(z + zg(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

in $\mathbb{C}[\![z]\!]$ is given by

$$\mathcal{L}_{\tilde{\varphi}} = \left\{g|g(z) = \tilde{g}_0(c_k z^k), c_k \in \mathbb{C}^\star\right\}.$$

The set $\mathcal{L}_{\tilde{\varphi}}$ is a subset of $\mathbb{C}[\![z^k]\!]$ and for every $c_k \in \mathbb{C}^\star$ equation (1.2) *has a unique solution g where $g(z) = c_k z^k + \ldots$.*

Conversely, if (1.2) *with* $\tilde{\varphi}(y) = y + d_2 y^2 + \dots$ *has a non constant solution* g*, then* $d_2 = k$*.*

Remark 1.3. In the situation where $w_0 = 1$ for every $c_k \in \mathbb{C}^\star$ the generalized Dhombres functional equation (1.1) $f(zf(z)) = \varphi(f(z))$ has a non constant solution

$$f(z) = 1 + \tilde{g}_0(c_k z^k)$$

where the formal series $\tilde{g}_0$ is computed in accordance to Theorem 1.2.

Theorem 1.2 is the analogue to Proposition 2 in [5], which we will denote by Proposition 1.4 and which is given by

Proposition 1.4. *Let* w_0 *be a complex number different from zero and no root of unity, and let the formal series* $\tilde{\varphi}$ *be given by* $\tilde{\varphi}(y) = w_0^k y^k + \dots$ *for a* $k \in \mathbb{N}$*. Then there exists a unique formal series* $\tilde{g}_0 \in \Gamma_1$ *such that the set of non constant soutions* g *of*

$$g(w_0 z + zg(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

in $\mathbb{C}[\![z]\!]$ *is given by*

$$\mathcal{L}_{\tilde{\varphi}} = \left\{ g | g(z) = \tilde{g}_0(c_k z^k), c_k \in \mathbb{C}^\star \right\}.$$

Then the set $\mathcal{L}_{\tilde{\varphi}}$ *is a subset of* $\mathbb{C}[\![z^k]\!]$*. Furthermore equation* (1.2) *has a unique solution* g *such that* $g(z) = c_k z^k + \dots$*, for every* $c_k \in \mathbb{C}^\star$*.*

In the case where $w_0 = 1$ it is not possible to use the methods which are developed in [5] to prove Proposition 2 because for $w_0 = 1$ there does not exist a Schröder function $S_{\tilde{\varphi}}$ such that we can write $\tilde{\varphi}(S_{\tilde{\varphi}}(z)) = S_{\tilde{\varphi}}(z)$. But with the approach we use in the proof of Theorem 1.2 we will see that it is also possible to prove Proposition 1.4.

2. Solutions f of the generalized Dhombres functional equation where $f(0) = 1$

In this section we prove Theorem 1.2 and therefore let us consider the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

where the solutions f fulfill $f(0) = 1$. Equation (1.1) becomes equivalent to equation (1.2), the transformed generalized Dhombres functional equation

$$g(z + zg(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

where we inserted $w_0 = 1$. Next we have to do some transformation steps. For the formal series g we substitute T^k and hence we get

$$T(z + zT(z)^k)^k = \tilde{\varphi}(T(z)^k).$$

Next we substitute $T^{-1}(z)$ for z and so

$$T(T^{-1}(z) + T^{-1}(z)z^k)^k = \tilde{\varphi}(z^k)$$

follows. From [5] it is known that $\tilde{\varphi}(z) = w_0^k z + d_2 z^2 + \dots$ and therefore with $w_0 = 1$ the formal series $\tilde{\varphi}$ is given as $\tilde{\varphi}(z) = z + \dots$ and hence $\tilde{\varphi}(z^k) = z^k + \dots \in \mathbb{C}[\![z^k]\!]$. Taking the $k - th$ root in the equation above leads to

$$T(T^{-1}(z) + T^{-1}(z)z^k) = \psi(z),$$

where we know that $\psi(z) = z + \dots$ and from Lemma 2 in [5] that $\psi(z) \in z\mathbb{C}[\![z^k]\!]$. We substitute U for T^{-1} where $U(z) = u_1 z + \dots$ and therefore we obtain the linear functional equation

$$(1 + z^k)U(z) = U(\psi(z)). \tag{2.1}$$

We write $U(z) = u_1 U^0(z) = u_1(z+\dots)$ and $U^0(z) = zU^*(z)$ where $U^*(z) = 1+\dots$. Hence we get

$$U(\psi(z)) = u_1 U^0(\psi(z)) = u_1 \psi(z) U^*(\psi(z)) = u_1 z \psi^*(z) U^*(\psi(z)),$$

where $\psi^*(z) = 1 + \dots \in \mathbb{C}[\![z^k]\!]$. With these transformations the linear functional equation (2.1) becomes equivalent to

$$(1 + z^k) u_1 z U^*(z) = u_1 z \psi^*(z) U^*(\psi(z)),$$

respectively

$$\frac{1 + z^k}{\psi^*(z)} U^*(z) = U^*(\psi(z)).$$

Taking the reciprocal of the series $\psi^*(z)$ leads to

$$\frac{1 + z^k}{\psi^*(z)} = \frac{1 + z^k}{1 + \beta z^k + \dots} = 1 + \alpha z^k + \dots \in \mathbb{C}[\![z^k]\!].$$

Hence we can introduce the formal logarithm and we write $LnU^*(z) =: Y(z)$ and $Ln\frac{1+z^k}{\psi^*(z)} =: A(z)$. For a more detailed discussion we have to write $\tilde{\varphi}(z) = z + d_2 z^2 + \dots$ which leads to

$$\begin{aligned} \tilde{\varphi}(z^k) &= z^k + d_2 z^{2k} + \dots + d_\nu z^{\nu k} + \dots \\ &= z^k (1 + d_2 z^k + \dots + d_\nu z^{(\nu-1)k} + \dots) \in \mathbb{C}[\![z^k]\!]. \end{aligned}$$

From the transformations we know that $\psi(z)^k = \tilde{\varphi}(z^k)$ where $\psi(z) = z + \dots$ and with the binomial series we obtain

$$\psi(z) = z(1 + d_2 z^k + \dots)^{\frac{1}{k}} = z(1 + \frac{d_2}{k} z^k + \dots) \in z\mathbb{C}[\![z^k]\!],$$

where $\psi^*(z) = 1 + \frac{d_2}{k} z^k + \dots \in \mathbb{C}[\![z^k]\!]$. Hence we know that the reciprocal of $\psi^*(z) = 1 + \frac{d_2}{k} z^k + \dots$ is given by

$$\psi^*(z)^{-1} = 1 - \frac{d_2}{k} z^k + \dots \in \mathbb{C}[\![z^k]\!].$$

So we get for the fraction $\frac{1+z^k}{\psi^*(z)}$ the expression

$$\frac{1 + z^k}{\psi^*(z)} = (1 + z^k)(1 - \frac{d_2}{k} z^k + \dots) = 1 + (1 - \frac{d_2}{k}) z^k + \dots \in \mathbb{C}[\![z^k]\!].$$

Finally for A we obtain

$$A(z) = Ln\frac{1 + z^k}{\psi^*(z)} = (1 - \frac{d_2}{k}) z^k + \dots \in \mathbb{C}[\![z^k]\!].$$

Therefore the formal series A is known and we are looking for a solution Y of the functional equation

$$A(z) + Y(z) = Y(\psi(z)). \tag{2.2}$$

Let us suppose that $\psi(z) = z$. Then $A(z) = 0$ and hence $\frac{1+z^k}{\psi^*(z)} = 1$ which is equivalent to $1 + z^k = \psi^*(z)$. From $\psi(z) = z\psi^*(z)$ follows that $z = z\psi^*(z)$ and this leads to $1 + z^k = 1$ and this is a contradiction. So we know that $\psi(z) \neq z$ and hence because the first coefficient of $\psi(z)$ is 1 we know that the formal series $\psi(z)$ is not linearizable.

We have to discuss the functional equation (2.2) and therefore we start with two lemmata where the first one deals with the homogenous equation associated with (2.2) and the second one with the inhomogenous equation (2.2).

Lemma 2.1. *Let Y and ψ be formal power series with $Y(z) = \beta_1 z + \beta_2 z^2 + \dots$ and $\psi(z) = 1 \cdot z + \dots$ where ψ is not linearizable. Then the functional equation*

$$Y(z) = Y(\psi(z))$$

has only the trivial solution.

Proof. We consider the equation $Y(z) = Y(\psi(z))$ where we know that ψ is no linear function, that means ψ is not equal to z, and hence it is possible to embed ψ in an iteration group $(\Theta_t)_{t\in\mathbb{C}}$ of type II. Therefore we get $Y(z) = Y(\Theta_t(z))$ for every $t \in \mathbb{C}$. This follows from Theorem 13 on page 10 in [3]. Taking the partial derivatives $\frac{\partial}{\partial t}$ on both sides we obtain

$$0 = \frac{\partial Y}{\partial X}|_{X=\Theta_t(z)} \frac{\partial \Theta_t(z)}{\partial t}.$$

For $t = 0$ the equation

$$0 = \frac{\partial Y}{\partial z} H(z),$$

where $H(z)$ denotes the generator of the iteration group, follows. The generator of a non trivial iteration group is always different from zero and hence $Y(z) = \mathit{const}$ which results in $Y(z) = 0$. $\square$

Therefore we know that the homogenous equation associated with (2.2) has only the trivial solution. Next we want to consider the structure of the solution of the inhomogenous equation.

Lemma 2.2. *Let $k \in \mathbb{N}$ and A be given such that $A(y) = a_k y^k + a_{2k} y^{2k} + \dots \in \mathbb{C}[\![y^k]\!]$. Then necessarily a solution Y of the functional equation*

$$A(y) + Y(y) = Y(\lambda(y)) \tag{2.3}$$

with $\lambda(y) = y + \dots \in y\mathbb{C}[\![y^k]\!]$, where λ is not linearizable, is an element of $\mathbb{C}[\![y^k]\!]$.

Proof. Let η be a root of unity primitive of order k. Substituting ηy for y in (2.3) leads to

$$A(\eta y) + Y(\eta y) = Y(\lambda(\eta y)).$$

Then $A(\eta y)$ is given by $A(\eta y) = a_k \eta^k y^k + a_{2k} \eta^{2k} y^{2k} + \dots$ and hence $A(\eta y) = A(y)$ because $\eta^{\nu k} = 1$ for every $\nu \geq 1$. For the series λ we obtain $\lambda(\eta y) = \eta y + \lambda_{k+1} \eta y^{k+1} + \dots$ and hence $\lambda(\eta y) = \eta\lambda(y)$. Therefore (2.3) is equivalent to

$$A(y) + Y(\eta y) = Y(\eta\lambda(y))$$

and hence $Y(\eta y)$ is a solution of (2.3). According to Lemma 2.1 it is known that a solution of (2.3) is uniquely determinded and so $Y(y) = Y(\eta y)$ and hence $Y(y) \in \mathbb{C}[\![y^k]\!]$. $\square$

With this background we can prove Theorem 1.2.

Proof of Theorem 1.2. We consider the functional equation

$$A(z)+Y(z)=Y(\psi(z)). \tag{2.2}$$

As a consequence of our considerations we can write

$$A(z)=Ln\frac{1+z^k}{\psi^*(z)}=(1-\frac{d_2}{k})z^k+\alpha_{2k}z^{2k}+\ldots+\alpha_{\nu k}z^{\nu k}+\ldots\in\mathbb{C}[\![z^k]\!]$$

where the coefficients $\alpha_{\nu k}$ for $\nu\geq 2$ are known, and according to Lemma 2.1 and Lemma 2.2 the series Y has the representation

$$Y(z)=\gamma_k z^k+\gamma_{2k}z^{2k}+\ldots\in\mathbb{C}[\![z^k]\!]$$

as well as

$$\psi(z)=z(1+\frac{d_2}{k}z^k+\ldots)\in z\mathbb{C}[\![z^k]\!].$$

Substituting $\psi(z)$ in $Y(z)$ leads to

$$\begin{aligned}Y(\psi(z))&=\gamma_k\psi(z)^k+\gamma_{2k}\psi(z)^{2k}+\ldots\\&=\gamma_k(z+\frac{d_2}{k}z^{k+1}+\ldots)^k+\gamma_{2k}(z+\frac{d_2}{k}z^{k+1}+\ldots)^{2k}+\ldots\\&=\gamma_k z^k+(k\gamma_k\frac{d_2}{k}+\gamma_{2k})z^{2k}+\ldots.\end{aligned}$$

Comparing the coefficients of z^k in (2.2) leads to $\alpha_k+\gamma_k=\gamma_k$ and hence $\alpha_k=0$. The coefficient α_k is given by $\alpha_k=1-\frac{d_2}{k}$ and hence from $1-\frac{d_2}{k}=0$ we get $d_2=k$. Therefore the formal series $\tilde{\varphi}$ has the form

$$\tilde{\varphi}(z)=z+kz^2+\ldots.$$

Comparing the coefficients of z^{2k} leads to

$$\gamma_k=\frac{\alpha_{2k}}{d_2},$$

and hence γ_k is uniquely determinded. Inductively we obtain

$$\gamma_{\nu k}=P_{vk}(\alpha_{(\nu+1)k},\gamma_k,\ldots,\gamma_{(\nu-1)k})$$

where P_{vk}, $\nu\geq 2$ denotes a polynomial. Therefore the series Y is uniquely determinded. In a next step we have to compute $\tilde{g}_0$. Reversing our calculations leads to

$$U^*(z)=exp(Y(z))\in\mathbb{C}[\![z^k]\!],$$

as well as

$$U^0(z)=zU^*(z)\in\Gamma_1^{(k)}=\mathbb{C}^{(k)}[\![z]\!]\cap\Gamma_1$$

and hence

$$U(z)=u_1U^0(z)\in\Gamma^{(k)}=\mathbb{C}^{(k)}[\![z]\!]\cap\Gamma.$$

We also used $T^{-1}=U$ and so $T^{-1}\in\Gamma^{(k)}$ and $T\in\Gamma^{(k)}$. The formal series T has the representation $T(z)=t_1z+t_{k+1}z^{k+1}+t_{2k+1}z^{2k+1}+\ldots$ and hence we write

$$T(z)=t_1z+t_{k+1}z^{k+1}+t_{2k+1}z^{2k+1}+\ldots=t_1z(1+\sum_{\nu\geq 1}\tilde{t}_\nu(t_1^kz^k)^\nu),$$

where $\tilde{t}_\nu=t_1^{-1}t_{\nu k+1}t_1^{-\nu k}$, for $\nu\geq 1$. Then

$$T(z)^k=t_1^kz^k(1+\sum_{\nu\geq 1}\tilde{t}_\nu(t_1^kz^k)^\nu)^k$$

and with $c_k = t_1^k$ and $t_1 = u_1^{-1}$ we obtain

$$g(z) = T(z)^k = c_k z^k (1 + \sum_{\nu \geq 1} \delta_\nu (c_k z^k)^\nu)$$

where the coefficients δ_ν, $\nu \geq 1$ are uniquely determinded. Then we can define

$$\tilde{g}_0(y) = y(1 + \sum_{\nu \geq 1} \delta_\nu y^\nu) \in \Gamma_1 \subseteq \mathbb{C}[\![y]\!],$$

and so we have $g(z) = \tilde{g}_0(c_k z^k)$. Then, like in the proof of Proposition 2 in [5], the transformation steps can be reversed and hence for every $c_k \in \mathbb{C}^\star$ the series $g(z)$ defined by $g(z) = \tilde{g}_0(c_k z^k)$ is a solution of the transformed generalized Dhombres functional equation (1.2). For a given $c_k \in \mathbb{C}^\star$ the solution g defined as $g(z) = \tilde{g}_0(c_k z^k)$ is unique. $\square$

This method leads us to a new proof of Proposition 2 of [5]. Therefore we have

Remark 2.3 (An alternative proof of Proposition 1.4). If we consider the case where w_0 is a complex number different from zero and also no root of unity, then it is always possible to transform the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

and hence also the transformed generalized Dhombres functional equation

$$g(w_0 z + zg(z)) = \tilde{\varphi}(g(z)) \tag{1.2}$$

to the linear functional equation

$$(w_0 + z^k)U(z) = U(\psi(z))$$

where $\psi(z) = w_0 z + \ldots$. Then it is possible to use the same methods which we used in this section to prove Theorem 1.2. Here we do not need a linearization function which is always needed in the known proofs. We have to mention that if $w_0 \in \mathbb{C}^\star \setminus \mathbb{E}$ we have to embed ψ in contrary to Lemma 2.1 in an iteration group of type I which we want to show in Lemma 2.4. The rest works similar.

Lemma 2.4. *Let Y and ψ be formal power series with $Y(z) = \beta_1 z + \beta_2 z^2 + \ldots$ and $\psi(z) = w_0 z + \ldots$ where $w_0 \in \mathbb{C}^\star \setminus \mathbb{E}$. Then the functional equation*

$$Y(z) = Y(\psi(z))$$

has only the trivial solution.

Proof. By induction we obtain $Y(z) = Y(\psi^n(z))$ for every $n \in \mathbb{N}$. Then we embed ψ in an iteration group $(\psi_t)_{t \in \mathbb{C}}$ of type I given by $\psi_t(z) = e^{\lambda t} z + \sum_{\nu \geq 2} Q_\nu(e^{\lambda t}) z^\nu$, $t \in \mathbb{C}$ where the Q_ν, $\nu \geq 2$ are polynomials and $\lambda = \log w_0$. Then we get

$$Y(\psi_t(z)) - Y(z) = \sum_{\nu \geq 1} R_\nu(e^{\lambda t}) z^\nu$$

where $R_\nu(e^{\lambda t})$ is a polynomial in $e^{\lambda t}$. For $t \in \mathbb{N}$ we obtain $0 = Y(\psi_t(z)) - Y(z)$ and hence we can also show

$$0 = Y(\psi_t(z)) - Y(z) = \sum_{\nu \geq 1} R_\nu(e^{\lambda t}) z^\nu$$

for all $t \in \mathbb{N}$: For every $\nu \geq 0$

$$R_\nu(e^{\lambda n}) = 0$$

is valid, for all $n \in \mathbb{N}$. The complex number w_0 is no root of unity and hence $\lambda \notin \mathbb{Q}$, therefore the sequence $(e^{\lambda n})_{n\in\mathbb{N}}$ has infinitely many values and so $R_\nu = 0$ for all $\nu \geq 0$. Finally for all $t \in \mathbb{C}$ we obtain

$$Y(\psi_t(z)) - Y(z) = \sum_{\nu\geq 1} R_\nu(e^{\lambda t})z^\nu = 0.$$

Then again we have

$$0 = \frac{\partial Y}{\partial X}|_{X=\psi_t(z)} \frac{\partial \psi_t(z)}{\partial t}.$$

Substituting $t = 0$ leads to

$$0 = \frac{\partial Y}{\partial z} H(z)$$

where $H(z)$ denotes the generator of the iteration group which is always different from zero and hence $Y(z) = 0$. □

At the end we want to give an example which shows that there are also local analytic solutions f of the generalized Dhombres functional equation where $f(0) = 1$.

Example 2.5. We consider the situation where $k = 1$ and according to Theorem 1.2 we choose ψ as $\psi(z) = z+z^2+z^3+\dots$ and this ψ is local analytic in a sufficiently small neighbourhood of zero. Solving the linear functional equation

$$(1 + z)U(z) = U(\psi(z)) \tag{2.1}$$

for $u_1 = 1$ respectively for $u_1 = -1$ gives us, after reversing our calculations

$$f(z) = 1 + z + z^2 + z^3 + z^4 + \dots$$

and

$$f(z) = 1 - z + z^2 - z^3 + z^4 + \dots$$

as non constant, local analytic solutions of the generalized Dhombres functional equation (1.1) $f(zf(z)) = \varphi(f(z))$.

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