LOCAL SOLUTIONS OF THE GENERALIZED DHOMBRES FUNCTIONAL EQUATION IN A NEIGHBOURHOOD OF INFINITY

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Abstract. We study the generalized Dhombres functional equation in the complex domain, we investigate solutions in a neighbourhood of infinity. After some transformations a characterization of those solutions which have a complex number different from zero or infinity as value is given. Moreover the convergence of the solutions and solutions represented as infinite product are figured out.

Introduction

The Dhombres functional equation was introduced in the year 1975 by J. Dhombres in [1]. In [2] Dhombres investigated solutions of the equation

$$f(xf(x)) = f(x)^2$$

in the real domain.
In the complex domain the generalized Dhombres functional equation is given by

$$f(zf(z)) = \varphi(f(z)), \tag{0.1}$$

where the function φ is known. This equation was first studied in [4]. **The aim of this paper is to investigate those solutions f of (0.1) with $f(\infty) = w_0$, where $w_0 \in \mathbb{C} \setminus \{0\}$ or $f(\infty) = \infty$.** The sections one and two deal with the case where $f(\infty) = w_0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. Therefore in the first section we start with some transformations which lead us to **new, not emerged equations** in the theory of generalized Dhombres functional equations before. After this transformations we distinguish solutions f of (0.1) belonging to various values of w_0 and we characterize the formal solutions for the different values of w_0. We also iterate some equations to obtain product solutions. The second section contains assertions on the convergence of the before computed solutions. In the last section we deal with those generalized Dhombres functional equations, respectively with those solutions of the equations which are determined by $f(\infty) = \infty$.

The definitions and facts which we now give are necessary for understanding the follwing. We define $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$, and when we talk about the complex number w_0 we always mean that $w_0 \in \mathbb{C}^\star$. By $\mathbb{E}$ we denote the set of the complex roots of one. We are interested in formal and local analytic power series solutions of the generalized Dhombres functional equation. Therefore by $\mathbb{C}[\![z]\!] = \{F : F(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \dots\}$ with $\beta_\nu \in \mathbb{C}$ for $\nu \geq 0$ we denote the ring of

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formal power series with usual addition, multiplication and substitution. For a series $F \in \mathbb{C}[\![z]\!]$, $F \neq 0$ the order of F is defined by

$$\operatorname{ord} F := \min\{\nu \in \mathbb{N} \colon \beta_\nu \neq 0\}$$

and one sets ∞ for the order of the trivial series. By Γ_1 we define the set of all formal power series starting with z, that means $\Gamma_1 = \big\{F \colon F \in \mathbb{C}[\![z]\!] \text{ and } F(z) \equiv z \pmod{z^2}\big\}$, which forms a group with respect to substitution. We will also often use the solution of the Schröder functional equation. We call a series $F(z) = \rho z + \dots$ linearizable if there exists a series S such that the Schröder functional equation

$$F(z) = S(\rho S^{-1}(z)) \tag{0.2}$$

holds. If F is linearizable, $F(z) = \rho z + \dots$, $\rho \neq 0$, it is known that there exists exactly one $S \in \Gamma_1$ and moreover if F is local analytic and linearizable then there exists a local analytic S such that (0.2) holds (see [7] and [9]).
If F is not linearizable we have the following theorem (see [8] and [10]).

Theorem 0.1. *Let the series F be given by $F(z) = \rho z + c_2 z^2 + \dots$ where $\rho = e^{2\pi i \frac{m}{k}}$, $k > 0$ and $\gcd(m,k) = 1$.*

1. *There exists a T with $T(z) = z + \dots$ such that $(T^{-1} \circ F \circ T)(z) = \rho z + \sum_{\nu \geq 1} \gamma_\nu z^{\nu k + 1} \in z\mathbb{C}[\![z^k]\!]$. The series $\rho z + \sum_{\nu \geq 1} \gamma_\nu z^{\nu k+1}$ is called semicanonical form of F.*
2. *The semicanonical form is (in general) not uniquely determined.*
3. *There exists exactely one transformation T of the form $T(z) = z + \sum_{\nu \not\equiv 1 \pmod{k}} t_\nu z^\nu$ such that $T^{-1} \circ F \circ T$ is a semicanonical form.*
4. *The series F is linearizable if and only if every semicanonical form of F is linear.*

1. Solutions f of $f(zf(z)) = \varphi(f(z))$ with $f(\infty) = w_0 \neq 0$

1.1. **Transformations.** We consider the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{0.1}$$

for a given φ. In this part of the article we want to transform equation (0.1) to an equation which is easier to handle. We also obtain a necessary and condition for the first coefficient of the series φ.
Let f be holomorphic in a neighbourhood of $z = \infty$ and let $f(\infty) = w_0 \in \mathbb{C}^\star$. Then we write $z = \frac{1}{u}$ where u is an element of a neighbourhood of zero and we get

$$f\left(\frac{1}{u} f\left(\frac{1}{u}\right)\right) = \varphi\left(f\left(\frac{1}{u}\right)\right).$$

We set $f\left(\frac{1}{u}\right) = \hat{f}(u)$, $\hat{f}$ is holomorphic in a neighbourhood of $u = 0$. We obtain

$$f\left(\frac{1}{u}\hat{f}(u)\right) = \varphi\left(\hat{f}(u)\right)$$

where we can rewrite the left hand side, and therefore we have

$$f\left(\frac{1}{\frac{u}{\hat{f}(u)}}\right) = \varphi\left(\hat{f}(u)\right).$$

We can use the definition of the function $\hat{f}$ once more on the left hand side, which leads to

$$\hat{f}\left(\frac{u}{\hat{f}(u)}\right) = \varphi\left(\hat{f}(u)\right).$$

Since $\hat{f}(0) = w_0$ it is possible to write $\hat{f}(u) = w_0 + g(u)$ where g is holomorphic in a neighbourhood of zero and $g(0) = 0$. Substituting this in the equation above leads to

$$w_0 + g\left(\frac{u}{w_0 + g(u)}\right) = \varphi(w_0 + g(u)).$$

By putting $u = 0$ we obtain $w_0 + g(0) = \varphi(w_0 + g(0))$ and hence $\varphi(w_0) = w_0$. Therefore we can represent φ as $\varphi(y) = w_0 + \tilde{\varphi}(y - w_0)$ with $\tilde{\varphi}(0) = 0$. So we get the transformed generalized Dhombres functional equation

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)). \tag{1.1}$$

This equation contains as expression the fraction $\frac{u}{w_0+g(u)}$ which is a **new outcome** in the theory of generalized Dhombres functional equations in the complex domain. Since we are interested in non constant solutions we write $g(u) = T(u)^k$ for a $k \in \mathbb{N}$ and ord $T = 1$. Substituting this depiction of g into (1.1) causes to

$$T\left(\frac{u}{w_0 + T(u)^k}\right)^k = \tilde{\varphi}(T(u)^k),$$

or if we substitute $T^{-1}(u)$ for u

$$T\left(\frac{T^{-1}(u)}{w_0 + u^k}\right)^k = \tilde{\varphi}(u^k).$$

Taking the k-th root leads to

$$T\left(\frac{T^{-1}(u)}{w_0 + u^k}\right) = \psi(u)$$

where $\psi(u)^k = \tilde{\varphi}(u^k)$, again this equation differs from the known theory because of the fraction in the brackets. We set $U = T^{-1}$ and so we obtain the linear functional equation

$$(w_0 + u^k)^{-1}U(u) = U(\psi(u)). \tag{1.2}$$

As a reference for linear functional equations we mention the book [3]. Next we want to determine the first coefficient of $\tilde{\varphi}$ and hence also of ψ. The order of T is one, we represent T as $T(u) = t_1 u + \dots$ with $t_1 \neq 0$. Then $T^{-1}(u) = \frac{1}{t_1}u + \dots$ and we get

$$\frac{T^{-1}(u)}{w_0 + u^k} = \frac{1}{w_0}\left(\frac{1}{t_1}u\right)\left(1 + \frac{u^k}{w_0}\right)^{-1} = \frac{1}{w_0}\frac{1}{t_1}u + \dots$$

which leads to

$$T\left(\frac{T^{-1}(u)}{w_0 + u^k}\right) = t_1\frac{1}{w_0}\frac{1}{t_1}u + \dots = \frac{1}{w_0}u + \dots = w_0^{-1}u + \dots,$$

or

$$T\left(\frac{T^{-1}(u)}{w_0 + u^k}\right)^k = w_0^{-k}u^k + \dots.$$

For $\tilde{\varphi}$ we write $\tilde{\varphi}(u) = d_1 u + \dots$ and so $\tilde{\varphi}(u^k) = d_1 u^k + \dots$. Finally $\tilde{\varphi}$ and ψ are given by $\tilde{\varphi}(u) = w_0^{-k} u + \dots$ and $\psi(u) = w_0^{-1} u + \dots$.
These considerations finish the subsection, in the following paragraphs we have to distinguish if w_0 is an element of $\mathbb{C}^\star \setminus \mathbb{E}$ or if w_0 is a root of one.

1.2. w_0 **no root of one.** First let w_0 be a complex number different from zero and not a root of one. Then it is known that for $\psi(u) = w_0^{-1} u + \dots$ there exists a function S_ψ such that $\psi(y) = S_\psi(w_0^{-1} S_\psi^{-1}(y))$ holds. The following theorem states a characterization of the solutions for this case.

Theorem 1.1. *Let $w_0 \in \mathbb{C}^\star \setminus \mathbb{E}$ and $\tilde{\varphi}$ be given by $\tilde{\varphi}(u) = w_0^{-k} u + \dots$. Then there exists a unique function $\tilde{g}_0$ with $\tilde{g}_0 \in \Gamma_1$ such that the non constant solutions g of*

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)) \tag{1.1}$$

in $\mathbb{C}[\![u]\!]$ are given by

$$\mathcal{L}_\psi = \left\{ g\colon g(u) = \tilde{g}_0(v_k u^k),\ v_k \in \mathbb{C}^\star \right\}.$$

$\mathcal{L}_\psi$ is a subset of $\mathbb{C}[\![u^k]\!]$ and for every $v_k \in \mathbb{C}^\star$ exists exactly one solution g with $g(u) \equiv v_k u^k \pmod{u^{k+1}}$.

Proof. From the previous subsection we know that the transformed generalized Dhombres functional equation (1.1) can be equivalently written as linear functional equation (1.2). In (1.2) we use the linearization of ψ and therefore (1.2) becomes equivalent to $(w_0 + S_\psi(u)^k)^{-1} U(S_\psi(u)) = U(S_\psi(w_0^{-1} u))$. Defining $V := U \circ S_\psi$ leads to

$$(w_0 + S_\psi(u)^k)^{-1} V(u) = V(w_0^{-1} u).$$

By using $V(u) = v_1 u + \dots = v_1 u V^\star(u)$ where $V^\star(u) = 1 + \dots$ we obtain

$$(1 + w_0^{-1} S_\psi(u)^k)^{-1} V^\star(u) = V^\star(w_0^{-1} u). \tag{1.3}$$

All terms start with one and hence we can use the formal logarithm. We set $A(u) = -\operatorname{Ln}(1 + w_0^{-1} S_\psi(u)^k)$ and $X(u) = \operatorname{Ln}(V^\star(u))$ and so the equation

$$A(u) + X(u) = X(w_0^{-1} u) \tag{1.4}$$

remains. We represent the series A and X as $A(u) = \alpha_1 u + \alpha_2 u^2 + \dots$ and $X(u) = \gamma_1 u + \gamma_2 u^2 + \dots$. By comparing the coefficients we obtain

$$\gamma_\nu = (w_0^{-\nu} - 1)^{-1} \alpha_\nu \tag{1.5}$$

for $\nu \in \mathbb{N}$. Hence the series X is uniquely determined. Then $V^\star$ is given by $V^\star(u) = \exp(X(u))$ and therefore

$$V(u) = v_1 u \exp(X(u)).$$

This leads to

$$U(u) = v_1 S_\psi(u) \exp(X(S_\psi(u)))$$

and hence $T(u) = v_1^{-1} u + \dots$. Now, because $g(u) = T(u)^k$ we can write

$$g(u) = \tilde{v}_1^k u^k + \dots = \tilde{v}_1^k u^k (1 + \dots) = \tilde{g}_0(v_k u^k). \tag{1.6}$$

Therefore there exists exactly one function $\tilde{g}_0$ such that all solutions g can be written in the form (1.6). $\square$

Remark 1.2. If we write $v_k u^k = y_k$ for $v_k \in \mathbb{C}^\star$, $k \in \mathbb{N}$, then all solutions f of (0.1) are given by

$$f(y_k) = w_0 + \tilde{g}_0\left(\frac{1}{y_k}\right).$$

Next we want to describe the solutions f of (0.1) as an infinite product. We already have investigated infinite products in [6]. The major difference to the case described in [6] is that we now consider the fixed value w_0 where $|w_0| > 1$ instead of $|w_0| < 1$. We prove the following lemma.

Lemma 1.3. *Let $|w_0| > 1$ and let ψ be given by $\psi(u) = w_0^{-1}u + \ldots$, local analytic for $|u| < r$ for some $r > 0$ and $\psi(u) = S_\psi(w_0^{-1}S_\psi^{-1}(u))$ where $S_\psi(u) = u + \ldots$. Then for every $v_1 \in \mathbb{C}^\star$ the function*

$$g(u) = \left(\left[v_1 S_\psi^{-1}(u)\prod_{\nu=0}^{\infty}\left(1 + w_0^{-1}\left(\psi^\nu(u)\right)^k\right)\right]^{[-1]}\right)^k$$

is a solution of

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)). \tag{1.1}$$

Proof. We iterate the equation

$$(1 + w_0^{-1}S_\psi(u)^k)^{-1}V^\star(u) = V^\star(w_0^{-1}u), \tag{1.3}$$

and therefore in the first iteration step we substitute $w_0^{-1}u$ for u, we obtain

$$(1 + w_0^{-1}S_\psi(w_0^{-1}u)^k)^{-1}(1 + w_0^{-1}S_\psi(u)^k)^{-1}V^\star(u) = V^\star(w_0^{-2}u).$$

By induction we get

$$\prod_{\nu=0}^{n-1}\left(1 + w_0^{-1}S_\psi(w_0^{-\nu}u)^k\right)^{-1}V^\star(u) = V^\star(w_0^{-n}u).$$

The series $V^\star$ is given by $V^\star(u) = 1 + \tilde{v}_1 u + \tilde{v}_2 u^2 + \ldots$ and hence

$$\lim_{n\to\infty}\left|V^\star\left(w_0^{-n}u\right)\right| = \lim_{n\to\infty}\left|1 + \tilde{v}_1 w_0^{-n}u + \tilde{v}_2 w_0^{-2n}u^2 + \ldots\right| = 1.$$

Therefore if equation (1.3) has a solution, we obtain

$$\prod_{\nu=0}^{\infty}\frac{1}{\left(1 + w_0^{-1}S_\psi(w_0^{-\nu}u)^k\right)^{-1}} = V^\star(u),$$

or

$$V^\star(u) = \prod_{\nu=0}^{\infty}\left(1 + w_0^{-1}S_\psi(w_0^{-\nu}u)^k\right)$$

as a local analytic solution. On the other hand, if we consider $\prod_{\nu=0}^{n-1}\left(1 + w_0^{-1}S_\psi(w_0^{-\nu}u)^k\right)$ we see, because S_ψ is local analytic and $|w_0| > 1$, that this expression is local analytic. But then also

$$V(u) = v_1 u\prod_{\nu=0}^{n-1}\left(1 + w_0^{-1}S_\psi(w_0^{-\nu}u)^k\right)$$

and because of $U = V \circ S_\psi^{-1}$,

$$U(u) = v_1 S_\psi^{-1}(u) \prod_{\nu=0}^{n-1} \left(1 + w_0^{-1} S_\psi(w_0^{-\nu} S_\psi^{-1}(u))^k\right)$$

are local analytic. An induction, for example see [6], shows that $\psi^\nu(u) = S_\psi(w_0^{-\nu} S_\psi^{-1}(u))$, and so

$$U(u) = v_1 S_\psi^{-1}(u) \prod_{\nu=0}^{n-1} \left(1 + w_0^{-1} \psi^\nu(u)^k\right).$$

The inverse of U with respect to substitution is the function T and $g(u) = T(z)^k$, so we get

$$g(u) = \left(\left[v_1 S_\psi^{-1}(u) \prod_{\nu=0}^{\infty} \left(1 + w_0^{-1} (\psi^\nu(u))^k\right)\right]^{[-1]}\right)^k,$$

where the function g is local analytic in some neighbourhood of zero. □

The product solutions of the generalized Dhombres functional equation for a convergent ψ are summarized in the following theorem.

Theorem 1.4. *Let $|w_0| > 1$ and ψ be given by $\psi(u) = w_0^{-1}u + \ldots$, local analytic for $|u| < r$ for some $r > 0$ and $\psi(u) = S_\psi(w_0^{-1} S_\psi^{-1}(u))$ where $S_\psi(u) = u + \ldots$. Then for every $v_1 \in \mathbb{C}^\star$*

$$f(u) = w_0 + \left(\left[v_1 S_\psi^{-1}\left(\frac{1}{u}\right) \prod_{\nu=0}^{\infty} \left(1 + w_0^{-1} \left(\psi^\nu\left(\frac{1}{u}\right)\right)^k\right)\right]^{[-1]}\right)^k$$

where $^{[-1]}$ denotes the inverse with respect to substitution, is a solution of

$$f(zf(z)) = \varphi(f(z)) \tag{0.1}$$

with $f(\infty) = w_0$.

Proof. The proof of this theorem immediately follows from Lemma 1.3. We only have to finish all of our previous transfomations. Therefore we recall that $\hat{f}(u) = w_0 + g(u)$ and $\hat{f}(u) = f\left(\frac{1}{u}\right)$. Hence $f(u) = \hat{f}\left(\frac{1}{u}\right)$ and so we obtain

$$f(u) = w_0 + \left(\left[v_1 S_\psi^{-1}\left(\frac{1}{u}\right) \prod_{\nu=0}^{\infty} \left(1 + w_0^{-1} \left(\psi^\nu\left(\frac{1}{u}\right)\right)^k\right)\right]^{[-1]}\right)^k.$$

□

We can also consider solutions represented as infinite product according to the weak topology. Therefore we have the following theorem, where the function ψ does not need to be convergent. For a definition and other useful properties of the weak topology we refer the reader to [6].

Theorem 1.5. *Let $|w_0| > 1$ and let ψ be given by $\psi(u) = w_0^{-1}u + \ldots$, and $\psi(u) = S_\psi(w_0^{-1} S_\psi^{-1}(u))$ where $S_\psi(u) = u + \ldots$. Then for every $v_1 \in \mathbb{C}^\star$ the function*

$$g(u) = \left(\left[v_1 S_\psi^{-1}(u) \prod_{\nu=0}^{\infty} \left(1 + w_0^{-1} (\psi^\nu(u))^k\right)\right]^{[-1]}\right)^k$$

is a solution of

$$g\left(\frac{u}{w_0+g(u)}\right)=\tilde{\varphi}(g(u)), \tag{1.1}$$

where the infinite product converges according to the weak limit

$$\prod_{\nu=0}^{\infty}\left(1+w_0^{-1}\left(\psi^{\nu}\left(u\right)\right)^k\right)=\mathrm{w}-\lim_{n\to\infty}\prod_{\nu=0}^{n}\left(1+w_0^{-1}\left(\psi^{\nu}\left(u\right)\right)^k\right).$$

The solutions f of (0.1) *depend on the parameter v_1 and are given by*

$$f(u)=w_0+\left(\left[v_1S_{\psi}^{-1}\left(\frac{1}{u}\right)\prod_{\nu=0}^{\infty}\left(1+w_0^{-1}\left(\psi^{\nu}\left(\frac{1}{u}\right)\right)^k\right)\right]^{[-1]}\right)^k.$$

Proof. Once more we consider the equation

$$(1+w_0^{-1}S_{\psi}(u)^k)^{-1}V^{\star}(u)=V^{\star}(w_0^{-1}u), \tag{1.3}$$

which we iterate, therefore we know that we get

$$\prod_{\nu=0}^{n-1}\left(1+w_0^{-1}S_{\psi}(w_0^{-\nu}u)^k\right)^{-1}V^{\star}(u)=V^{\star}(w_0^{-n}u).$$

The series $V^{\star}$ is given by $V^{\star}(u)=1+\tilde{v}_1u+\tilde{v}_2u^2+\ldots$ and hence

$$\mathrm{w}-\lim_{n\to\infty}V^{\star}\left(w_0^{-n}u\right)=\lim_{n\to\infty}(1+\tilde{v}_1w_0^{-n}u+\tilde{v}_2w_0^{-2n}u^2+\ldots)=1.$$

Therefore, if equation (1.3) has a solution, again we obtain

$$V^{\star}(u)=\prod_{\nu=0}^{\infty}\left(1+w_0^{-1}S_{\psi}(w_0^{-\nu}u)^k\right)$$

as solution. Otherwise we have to show that $\prod_{\nu=0}^{\infty}\left(1+w_0^{-1}S_{\psi}(w_0^{-\nu}u)^k\right)$ is convergent according to the weak topology. It is sufficient, see [6], to show that the weak limit of $\sum_{\nu=0}^{n}\mathrm{Ln}\left(1+w_0^{-1}S_{\psi}(w_0^{-\nu}u)^k\right)$ exists, for $n\to\infty$. This follows immediately form the representation

$$\mathrm{Ln}\left(1+w_0^{-1}S_{\psi}(w_0^{-\nu}u)^k\right)=\sum_{l=1}^{\infty}(-1)^{l-1}\frac{\left(1+w_0^{-1}S_{\psi}(w_0^{-\nu}u)^k\right)^l}{l}, \tag{1.7}$$

because we can order the right hand side of (1.7) with respect of u^m, $m\in\mathbb{N}$ and we observe, that to the coefficient of every u^m there belongs a term of the form $w_0^{-\mu}$ with $\mu\geq m$. Therefore the right hand side is convergent according to the weak topology, because every coefficient belongs to a geometric series. Reversing our transformations lead to the above given functions g and f. □

1.3. w_0 **is a primitive root of one of order** l**.** For the second case let w_0 be a primitive root of unity of order $l\in\mathbb{N}$. Then also w_0^{-1} is a root of unity of order l. From the linear functional equation (1.2) the equivalent expression

$$\frac{w_0^{-1}u(1+w_0^{-1}u^k)^{-1}}{\psi(u)}U^{\star}(u)=U^{\star}(\psi(u))$$

or

$$\frac{1}{(1+w_0^{-1}u^k)\psi^{\star}(u)}U^{\star}(u)=U^{\star}(\psi(u))$$

follows, where $U^\star(u) = 1 + \dots$ and $\psi^\star(z) = 1 + \dots$. Then we use the formal logarithm, we write $A(u) = \operatorname{Ln} \frac{1}{(1+w_0^{-1}u^k)\psi^\star(u)}$ and $X(u) = \operatorname{Ln} U^\star(u)$. Therefore we obtain

$$A(u) = X(\psi(u)) - X(u). \tag{1.8}$$

Next we distinguish the cases where ψ is linearizable and where it is not. Note that the function $\tilde{\varphi}$ is linearizable if and only if the function ψ is.
Let ψ be linearizable. Then there exists a minimal $m \in \mathbb{N}$ such that $\psi^m(u) = u$. This is clear, because since ψ is linearizable there exists a unique function S_ψ, $S_\psi(u) = u + \dots$ such that

$$\psi(u) = S_\psi(w_0^{-1} S_\psi^{-1}(u)) \tag{1.9}$$

holds. Then if we iterate this expression l times we get $\psi^l(u) = S_\psi(w_0^{-l} S_\psi^{-1}(u)) = S_\psi(S_\psi^{-1}(u)) = u$ and hence l is the minimal natural number such that $\psi^m(u) = u$. holds.

Theorem 1.6. *Let w_0 be a root of one primitive of order $l \geq 2$ and $\psi(u) = w_0 u + \dots$, and let ψ be linearizable with $\psi(u) = S_\psi(w_0 S_\psi^{-1}(u))$. Then the equation*

$$A(u) = X(\psi(u)) - X(u) \tag{1.8}$$

has a solution if and only if a certain infinite system of algebraic relations for the coefficients of S_ψ is satisfied. This infinite system is denoted by (1.12)*. If this is the case the general solution of*

$$(w_0 + u^k)^{-1} U(u) = U(\psi(u)) \tag{1.2}$$

is given by

$$U(u) = v_1 S_\psi^{-1}(u) exp(X(u))$$

where $X = Y \circ S_\psi^{-1}$ is given by

$$X(u) = Y(S_\psi^{-1}(u)) = \left[\sum_{\substack{\nu \geq 1 \\ \nu \not\equiv 0 (mod\ l)}} \frac{\tilde{s}_\nu}{w_0^{-\nu} - 1} u^\nu + \sum_{\nu \geq 1} \delta_{\nu l} u^{\nu l} \right]$$

where the terms $\tilde{s}_\nu$ for $\nu \geq 1$ and $\nu \not\equiv 0 (mod\ l)$ are computed from the coefficients of S_ψ and where the coefficients $\delta_{\nu l}$ for $\nu \geq 1$ are arbitrary.

Proof. We iterate (1.8) by substituting $\psi(u)$ for u. This leads to

$$A(\psi(u)) = X(\psi(u)) - A(u) - X(u),$$

by induction we obtain

$$\sum_{j=0}^{l-1} A(\psi^j(u)) = X(\psi^l(u)) - X(u),$$

or

$$\sum_{j=0}^{l-1} A(\psi^j(u)) = 0.$$

Here we substitute the representation of ψ given by the linearization function (1.9). Hence we obtain

$$\sum_{j=0}^{l-1} A(S_\psi(w_0^{-j}u)) = 0. \tag{1.10}$$

We define $Z(u) = \sum_{j=0}^{l-1} A(S_\psi(w_0^{-j}u))$, then

$$Z(w_0^{-1}u) = Z(u).$$

Therefore $Z(u) \in \mathbb{C}[\![u^l]\!]$ and we can write

$$Z(u) = \sum_{\substack{\nu\equiv 0 \pmod l \\ \nu\geq 1}} \gamma_\nu u^\nu. \tag{1.11}$$

As a consequence of (1.10) and (1.11) the series $S_\psi(u) = u+s_2u^2+\ldots = \sum_{\nu\geq 1} s_\nu u^\nu$ has to fulfill the polynomial system

$$s_{\mu l} = P_\mu(s_1, \ldots, s_{\mu l-1}), \quad \mu \geq 1. \tag{1.12}$$

In the last step we substitute the representation of ψ, namely (1.9) in (1.8), then we get

$$A(S_\psi(u)) = X(S_\psi(w_0^{-1}u)) - X(S_\psi(u)). \tag{1.13}$$

By $B(u)$ we address the series $A(S_\psi(u))$ and by $Y(u)$ the series $X(S_\psi(u))$, we write $B(u) = \sum_{\nu\geq} \beta_\nu u^\nu$ and $Y(u) = \sum_{\nu\geq 1} \gamma_\nu u^\nu$. From (1.13) we obtain

$$B(u) = Y(w_0^{-1}u) - Y(u). \tag{1.14}$$

Equation (1.14) leads to the special solution

$$\gamma_\nu = \frac{\beta_\nu}{w_0^{-\nu} - 1}$$

for $\nu \geq 1$, $\nu \neq \mu l$ for $l \geq 1$. Then the general solution of (1.8) is given by

$$X(u) = Y(S_\psi^{-1}(u)) = \left[\sum_{\substack{\nu\geq 1 \\ \nu\not\equiv 0 (mod\ l)}} \frac{\tilde{s}_\nu}{w_0^{-\nu} - 1} u^\nu + \sum_{\nu\geq 1} \delta_{\nu l} u^{\nu l}\right].$$

□

Let ψ be non linearizable, then as mentioned in the introduction, there exists a unique function $T \in \mathbb{C}[\![u]\!]$, $T(u) = u + \ldots$ such that

$$T^{-1}(\psi(T(u))) = w_0^{-1}u + \sum_{\nu\geq m} \gamma_\nu u^{\nu l+1}, \tag{1.15}$$

where $N_\psi(u) = w_0^{-1}u + \sum_{\nu\geq m} \gamma_\nu u^{\nu l+1}$, $\gamma_m \neq 0$ is called a semicanonical form of ψ and $m \in \mathbb{N}$.

Theorem 1.7. *Let w_0 be a root of one primitive of order $l \geq 1$ and let $\psi(z) = w_0 u + \ldots$ be not linearizable, and hence $\psi(u) = (T^{-1} \circ N \circ T)(u)$ where N is a semicanonical form of ψ. Then the equation*

$$(w_0 + u^k)^{-1}U(u) = U(\psi(u)) \tag{1.2}$$

has a unique solution, for a given v_1 if and only if a certain system of finitely many polynomial relations for the coefficients of T is satisfied.

Proof. Again we consider the equation

$$A(u) = X(\psi(u)) - X(u) \tag{1.8}$$

where we substitute the representation $T(N_\psi(T^{-1}(u)))$ for $\psi(u)$, then we get

$$A(u) = X(T(N_\psi(T^{-1}(u)))) - X(u)$$

and hence

$$A(T(u)) = X(T(N_\psi(u))) - X(T(u)).$$

We define $B(u) = A(T(u)) = \sum_{\nu\geq 1} \beta_\nu u^\nu$ and $Y(u) = X(T(u)) = \sum_{\nu\geq 1} \delta_\nu u^\nu$. Therefore the equation above is equivalent to

$$B(u) = Y(N_\psi(u)) - Y(u). \tag{1.16}$$

By Lemma 1 in [10] we can compute the series $Y(N_\psi(u)) - Y(u)$, it starts with

$$\begin{aligned} Y(N_\psi(u)) - Y(u) = \delta_1(w_0^{-1} - 1)u + \delta_1(w_0^{-2} - 1)u^2 + \ldots + \delta_l(w_0^{-l} - 1)u^l + \ldots \\ + (\delta_1\gamma_{(m+1)l} + \delta_l l w_0^{-1}\gamma_{ml+1} + \delta_{(m+1)l} - \delta_{(m+1)l})u^{(m+1)l} + \ldots. \end{aligned}$$

By comparing the coefficients we obtain $\delta_1 = \frac{\beta_1}{w_0^{-1}-1}$, $\delta_2 = \frac{\beta_2}{w_0^{-2}-1}$ and so on. If we compare the coefficients of z^l we get $\beta_l = \delta_l - \delta_l$. Since this β_l originates from the coefficients of T this is the first polynomial relation for the coeffcients of T which has to be fufilled. We obtain the other $l-1$ polynomial relations if we compare the coefficients of $z^{2l}, \ldots, z^{ml}$. The coefficient δ_l can be determined by comparing the coefficients of $u^{(m+1)l}$, we obtain $\delta_l = \frac{\beta_{(m+1)l} - \delta_1\gamma_{(m+1)l}}{l w_0^{-1}\gamma_{ml+1}}$. By induction the coefficients of Y are uniquely determined. By reversing our calculations we obtain the claim. □

To obtain solutions of the generalized Dhombres functional equation (0.1) we have to reverse the transformation steps from (0.1) to the linear functional equation (1.2). So the solutions U which we compute in Theorem (1.6) and (1.7) can be transformed back such that they are solutions of (0.1).

2. Local analytic solutions f of $f(zf(z)) = \varphi(f(z))$ with $f(\infty) = w_0 \neq 0$

In this section we want to discuss the convergence of the solutions which are not represented as a product. We start with the case where $|w_0| \neq 0, 1$. Then we have the following theorem.

Theorem 2.1. *Let $|w_0| \neq 0, 1$ and let $\tilde{\varphi}(u) = w_0^{-k}u + \ldots$ be local analytic for $|u| < r$, $r > 0$. Then all solutions g of*

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)) \tag{1.1}$$

are local analytic in a sufficiently small neighbourhood of zero. Equally all solutions f with $f(\infty) = w_0$ of

$$f(zf(z)) = \varphi(f(z)) \tag{0.1}$$

are local analytic in a neighbourhood of infinity.

Proof. Let $|w_0| \neq 0, 1$ and let $\tilde{\varphi}(u) = w_0^{-k}u + \dots$ be local analytic for $|u| < r$, $r > 0$. Then also $\psi(u)$ is local analytic in a sufficiently small neighbourhood of zero, because $\psi(u) = \tilde{\varphi}(u^k)^{\frac{1}{k}}$. If ψ is local analytic, then also the Schröder function S_ψ in the representation

$$\psi(u) = S_\psi(w_0^{-1}S_\psi^{-1}(u))$$

is local analytic. In the proof of Theorem 1.1 we obtain the equation

$$A(u) + X(u) = X(w_0^{-1}u) \tag{1.4}$$

where the series $A(u) = \alpha_1 u + \alpha_2 u^2 + \dots$ is known. From this proof we also know that the coefficients of the series $X(u) = \gamma_1 u + \gamma_2 u^2 + \dots$ compute as follows

$$\gamma_\nu = (w_0^{-\nu} - 1)^{-1}\alpha_\nu \tag{1.5}$$

for $\nu \in \mathbb{N}$. Since the absolute value of w_0 is not one there exists a number $C > 0$ such that

$$\left|w_0^{-\nu} - 1\right|^{-1} \leq C. \tag{2.1}$$

for all $\nu \in \mathbb{N}$. Then we have

$$|\gamma_\nu| \leq C\,|\alpha_\nu|$$

for all $\nu \in \mathbb{N}$. Hence X and also $V(u) = v_1 u \exp(X(u))$, which depends on the arbitrary parameter v_1, are local analytic in a sufficiently small neighbourhood of zero. By using elementary facts about holomorphic functions we obtain that

$$g(u) = T(u)^k = \left(U^{-1}(u)\right)^k = \left[(v_1 S_\psi(u)\exp(X(S_\psi(u))))^{-1}\right]^k$$

is local analytic in a sufficiently small neighbourhood of zero, for every parameter $v_1 \in \mathbb{C} \setminus \{0\}$. The function f is defined as $f(z) = w_0 + g\left(\frac{1}{z}\right)$ and therefore f is local analytic in a sufficiently small neighbourhood of infinity. □

In the next theorem we consider the case where w_0 is a Siegel number, that is a number which can be represented by $w_0 = e^{2\pi i\alpha}$ with $\alpha \in [0, 1)$ and for this α the following is true. There exist ϵ, μ greater than zero such that for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$

$$|n\alpha - m| > \epsilon n^{-\mu} \tag{2.2}$$

holds. This definition also implies that $|w_0| = 1$. First we want to show that the inequality (2.2) also holds for w_0^{-1}.

Remark 2.2. If w_0 is a Siegel number, then (2.2) also holds for w_0^{-1}.

Proof. We write $w_0^{-1} = e^{2\pi i(-\alpha)}$ and then we get

$$|n(-\alpha) - m| = |-n\alpha - m| = |-1||n\alpha + m| > \epsilon n^{-\mu}$$

since (2.2) holds for all $m \in \mathbb{Z}$. □

After showing that this remark hold, we see that the proof of the following theorem is essentially the same as the proof in Section 3 in [5].

Theorem 2.3. *Let w_0 be a Siegel number and let $\tilde{\varphi}(u) = w_0^{-k}u + \dots$ be local analytic for $|u| < r$, $r > 0$. Then all solutions g of*

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)) \tag{1.1}$$

are local analytic in a sufficiently small neighbourhood of zero. Equally all solutions f with $f(\infty) = w_0$ of

$$f(zf(z)) = \varphi(f(z)) \tag{0.1}$$

are local analytic in a neighbourhood of infinity.

Proof. Like in the previous proof we consider again the coefficient representation of a solution $X(u) = \gamma_1 u + \gamma_2 u^2 + \ldots$ of (1.4)

$$\gamma_\nu = (w_0^{-\nu} - 1)^{-1}\alpha_\nu \tag{1.5}$$

for $\nu \in \mathbb{N}$. Then according to the theory of Siegel numbers, which is given in [9], there exists a $\mu > 0$ such that

$$\left|w_0^{-1} - 1\right|^{-1} < (2\nu)^\mu$$

for $n \in \mathbb{N}$ holds. Then we have

$$|\gamma_\nu| < (2\nu)^\mu \, |\alpha_\nu|$$

for $n \in \mathbb{N}$. For $\delta > 1$ there exists a $n_0 \in \mathbb{N}$ such that $(2\nu)^\mu < \delta^\nu$ for $\nu > n_0$. Since the $\alpha_\nu's$ belong to a convergent series there exists a $\beta > 0$ such that for all $\nu > n_0$

$$|\alpha_\nu| < \beta^\nu.$$

Then we have

$$|\gamma_\nu| < (\delta\beta)^\nu$$

for all $\nu > n_0$ and hence the series X is local analytic in a sufficiently small neighbourhood of zero. The remaining reversing steps are the same as in the proof of the previous theorem. □

The proof of the last theorem in this section depends on [7]. There it is shown that if $\psi(u) = w_0^{-1}u + \ldots$, where w_0^{-1} is a root of one and ψ is linearizable, then there exists a convergent series S_ψ with $\psi(u) = S_\psi(w_0^{-1}S_\psi^{-1}(u))$. We have the following theorem which has its analogue in [10].

Theorem 2.4. *Let w_0 be a root of one and let $\tilde{\varphi}(u) = w_0^{-k}u + \ldots$ be local analytic for $|u| < r$, $r > 0$. Then there exist local analytic solutions g of*

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)) \tag{1.1}$$

in a sufficiently small neighbourhood of zero. Then there also exist solutions f with $f(\infty) = w_0$ of

$$f(zf(z)) = \varphi(f(z)) \tag{0.1}$$

which are local analytic in a neighbourhood of infinity.

Proof. Let the series $\tilde{\varphi}$ be local analytic, then also the series ψ is local analytic. We investigate the equation

$$B(u) = Y(w_0^{-1}u) - Y(u) \tag{1.14}$$

from the proof of Theorem 1.6. If ψ is local analytic, then also the series B on the left hand side of (1.14) is local analytic, which is a consequence of the representation

$$B(u) = A(S_\psi(u)) = \mathrm{Ln}\left(\frac{1}{(1 + w_0^{-1}S_\psi(u)^k)\psi^\star(u)}\right).$$

It is clear that $\psi^\star$ is convergent, from [7] we know that there exists a convergent series S_ψ. But then also the solution

$$Y(u) = \sum_{\substack{\nu \geq 1 \\ \nu \not\equiv 0 (mod\ l)}} \frac{\tilde{s}_\nu}{w_0^{-\nu} - 1} u^\nu + \sum_{\nu \geq 1} \delta_{\nu l} u^{\nu l}$$

is local analytic because the first sum is obtained from the composition of local analytic series and in the second sum we can choose the coefficients such that Y is local analytic. The claim of the theorem follows from reversing the transformations. □

3. Formal and local analytic solutions f of $f(zf(z)) = \varphi(f(z))$ with $f(\infty) = \infty$

In this section we want to consider solutions f of the generalized Dhombres functional equation (0.1) where $f(\infty) = \infty$. We will immediately see that after we apply some transformations we get a well known equation where the solutions which we want to determine have a very useful fixed point.
Let u be given in a neighbourhood of zero, then we use again $z = \frac{1}{u}$. Hence (0.1) becomes equivalent to

$$f\left(\frac{1}{u} f\left(\frac{1}{u}\right)\right) = \varphi\left(f\left(\frac{1}{u}\right)\right),$$

or

$$f\left(\frac{1}{u} \frac{1}{f\left(\frac{1}{u}\right)}\right) = \varphi\left(\frac{1}{f\left(\frac{1}{u}\right)}\right).$$

Then we define $h(u) = \frac{1}{f\left(\frac{1}{u}\right)}$ and so we have $h(0) = \frac{1}{\infty} = 0$. Substituting h in the equation above leads to

$$f\left(\frac{1}{u} \frac{1}{h(u)}\right) = \varphi\left(\frac{1}{h(u)}\right).$$

We can transform this equation to

$$\frac{1}{\frac{1}{f(uh(u))}} = \varphi\left(\frac{1}{h(u)}\right)$$

and hence we obtain

$$\frac{1}{h(uh(u))} = \varphi\left(\frac{1}{h(u)}\right).$$

In this equation we set $u = 0$ and so we get $\frac{1}{h(0)} = \varphi\left(\frac{1}{h(0)}\right)$ and therefore $\varphi(\infty) = \infty$. So we can define $\tilde{\varphi}(u) = \frac{1}{\varphi\left(\frac{1}{u}\right)}$ which is the same as $\frac{1}{\tilde{\varphi}(u)} = \varphi\left(\frac{1}{u}\right)$ and we have $\tilde{\varphi}(0) = 0$. Finally we obtain

$$\frac{1}{h(uh(u))} = \frac{1}{\tilde{\varphi}(h(u))}$$

or

$$h(uh(u)) = \tilde{\varphi}(h(u)). \tag{3.1}$$

Equation (3.1) is a generalized Dhombres functional equation for a given $\tilde{\varphi}$ and for an unknown function h with $h(0) = 0$. The case where $h(0) = 0$ is completely solved in [4]. We can use the same techniques as in [4], namely the transformations

$h(u) = T(u)^k$ and $\tilde{\varphi}(u^k) = \psi(u)^k$ to obtain the generalized Böttcher functional equation

$$u^k U(u) = U(\psi(u)).$$

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