ON A FUNCTIONAL-DIFFERENTIAL EQUATION OF A. F. BEARDON AND FUNCTIONAL-DIFFERENTIAL EQUATIONS OF BRIOT-BOUQUET TYPE

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ABSTRACT. We look for solutions of the functional-differential equation $f(\varphi(z)) = a(z)f(z)f'(z)$ for given series φ and a. We show that all formal solutions f of this equation are local analytic. In order to do this we transform the equation to a special type of Briot-Bouquet differential equation.

1. Introduction and Main Theorems

In [1] A. F. Beardon studied entire solutions $f(z)$ with $f(0) = 0$, $f \neq 0$ of the functional-differential equation

$$f(kz) = kf(z)f'(z) \tag{1.1}$$

when $|k| > 1$. To obtain such solutions, for appropriate values of k, first local analytic solutions are constructed and then, by means of (1.1) these local analytic solutions are holomorphically continued to entire ones.
Another method to find the local analytic solutions f of (1.1) with $f(0) = 0$, $f \neq 0$ consists of transforming (1.1) equivalently to a Briot-Bouquet type functional-differential equation which is similar to the classical Briot-Bouquet differential equation (see [3], [4] and [5]) with respect to the existence of formal solutions and to the proof of their convergence using appropriate majorants and an implicit function problem. This approach also works in case of equations

$$f(\varphi(z)) = a(z)f(z)f'(z) \tag{1.2}$$

where $\varphi(z) = kz + \varphi_2 z^2 + \ldots$, with $|k| > 1$, k is a Siegel number or $|k| < 1$, and $a(z) = a_0 + a_1 z + \ldots$, with $a_0 \neq 0$. φ and a are supposed to converge in a neighbourhood of 0. For the definition of a Siegel number we refer the reader to [6], page 166. Nevertheless we want to mention that $\rho = e^{2\pi i\alpha}$ with $0 \leq \alpha < 1$ and α irrational, is a Siegel number, if for α there exist $\epsilon > 0$ and $\mu > 0$ such that for all $n \in \mathbb{N}$ and all $m \in \mathbb{Z}$

$$|n\alpha - m| > \epsilon n^{-\mu}$$

holds.

Our procedure is as follows:
First we transform (1.2) as an application of Schröder's equation to an equation of the form

$$h(kz) = b(z)h(z)h'(z) \tag{1.3}$$

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with $h(0) = 0$, $h \neq 0$ and $b(z) = a_0 + b_1 z + \ldots$. The next step is to transform (1.3) by putting $h(z) = \rho z + z g(z)$, $g(0) = 0$ to an equivalent Briot-Bouquet type functional-differential equation

$$zg'(z) = g(kz) - 2g(z) + Q_1(g(z), g(kz), z) + Q_2(z) \tag{1.4}$$

with power series $Q_1(u, v, w)$ of order ord $Q_1 \geq 2$, where Q_1 does not contain any term cw^ν, $c \neq 0$ for $\nu \in \mathbb{N}$, and ord $Q_2 \geq 1$ and $g(z) = \sum_{\nu \geq 1} \gamma_\nu z^\nu$. Then we present a complete description of the formal solutions of (1.4) and hence of (1.2). After that the main task is to show that all formal solutions are convergent. Equations of the form (1.4) show a great similarity to the classical Briot-Bouquet differential equations. Even more general equations than (1.2) can be solved by these methods.

We call S, $S(z) = z + \ldots$ the linearization function of φ if $\varphi(z) = S^{-1}(kS(z))$. It is well-known that S is convergent and uniquely determined, if $|k| \neq 1$ or if k is a Siegel number (see [6]).

We prove the following two theorems. The first one describes the formal solutions of equation (1.2).

Theorem 1.1 (Formal solutions)**.** *Let the functional-differential equation*

$$f(\varphi(z)) = a(z) f(z) f'(z) \tag{1.2}$$

be given, where $\varphi(z) = kz + d_2 z^2 + \ldots$, φ linearizable (as formal series which is always possible if k is not a root of unity), and $a(z) = a_0 + a_1 z + \ldots$ with $a_0 \neq 0$ are given and let S be the linearization function of φ.

1. *If we assume that for all $\nu \in \mathbb{N}$ the relation $\nu + 2 - k^\nu \neq 0$ holds, then*
   - (a) *if $a(z) - a_0 = 0$, there exists a uniquely determined formal solution f of (1.2). This f is given by $f(z) = k a_0^{-1} z$,*
   - (b) *if $a(z) - a_0 \neq 0$, there exists a uniquely determined formal solution f of (1.2). This f is given by $f(z) = k a_0^{-1} S(z) + S(z) g(S(z))$ where g is a solution of (1.4).*
2. *If there exists a $\nu_0 \in \mathbb{N}$ such that $\nu_0 + 2 - k^{\nu_0} = 0$, then there exists a formal solution f of (1.2) if and only if a certain polynomial relation between the coefficients of Q_1 and Q_2 and therefore between the coefficients of a is fulfilled. Let this condition hold. Then for each arbitrary γ_{ν_0} there exists exactly one solution g of (1.4), $g(z) = \gamma_1 z + \ldots + \gamma_{\nu_0} z^{\nu_0} + \ldots$. In this case f is given by $f(z) = k a_0^{-1} S(z) + S(z) g(S(z))$.*

It is obvious that the following remark holds.

Remark 1.2. If k is a Siegel number or $|k| < 1$ then there exists no $\nu_0 \in \mathbb{N}$ such that $\nu_0 + 2 - k^{\nu_0} = 0$.

After computing the formal solutions we want to prove the convergence.

Theorem 1.3. *Let the functional-differential equation*

$$f(\varphi(z)) = a(z) f(z) f'(z) \tag{1.2}$$

be given, where $\varphi(z) = kz + d_2 z^2 + \ldots$ with $|k| > 1$, k is a Siegel number or $|k| < 1$ and $a(z) = a_0 + a_1 z + \ldots$ with $a_0 \neq 0$ are given local analytic functions. Then all formal solutions f, $f(z) = c_1 z + c_2 z^2 + \ldots$ with $f(0) = 0$ are local analytic.

For a better understanding we give the following definition (see [2]).

Definition 1.4. By $\mathbb{C}[\![z]\!] = \{F\colon F(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \dots\}$ with $\beta_\nu \in \mathbb{C}$ for $\nu \geq 0$ we denote the ring of formal power series with usual addition, multiplication and substitution. For a series $F \in \mathbb{C}[\![z]\!]$, $F \neq 0$ the order of F is defined by

$$\operatorname{ord} F := \min\{\nu \in \mathbb{N}\colon \beta_\nu \neq 0\}$$

and one sets ∞ for the order of the trivial series. By Γ_1 we define the set $\Gamma_1 = \{F\colon F \in \mathbb{C}[\![z]\!] \text{ and } F(z) \equiv z \pmod{z^2}\}$, which forms a group with respect to substitution.

The following remark will become important for our further transformations.

Remark 1.5. Let f be a formal solution of (1.2) with $f(0) = 0$, $f \neq 0$. Then $f(z) = \rho z + \dots$, where $\rho \neq 0$.

Proof. We have $f(0) = 0$ and $f \neq 0$ and hence we assume that $\operatorname{ord} f = m \in \mathbb{N}$. Then $\operatorname{ord} f' = m - 1$. We obtain $\operatorname{ord} f(\varphi(z)) = m$ for the left hand side of (1.2). The order of a is zero and so the order of the right hand side of (1.2) is $0+m+m-1$. Comparing both sides leads to $m = 1$ and hence $f(z) = \rho z + \dots$ with $\rho \neq 0$. □

2. Transformation to a functional-differential equation of Briot-Bouquet type

Since $|k| > 1$, k is a Siegel number or $|k| < 1$, there exists a convergent series $S \in \Gamma_1$ such that $\varphi(z) = S^{-1}(kS(z))$. So equation (1.2) becomes equivalent to

$$f(S^{-1}(kS(z)) = a(z)f(z)f'(z).$$

Substituting $S^{-1}(z)$ for z leads to

$$f(S^{-1}(kz)) = a(S^{-1}(z))f(S^{-1}(z))f'(S^{-1}(z)).$$

We have $\left(f \circ S^{-1}\right)'(z) = f'(S^{-1}(z))(S^{-1})'(z)$, and hence we obtain

$$f(S^{-1}(kz)) = a(S^{-1}(z))f(S^{-1}(z))\frac{\left(f \circ S^{-1}\right)'(z)}{(S^{-1})'(z)}.$$

Define h by $h = f \circ S^{-1}$ results in

$$h(kz) = b(z)h(z)h'(z) \tag{1.3}$$

where b is given by $b(z) = \frac{a(S^{-1}(z))}{(S^{-1})'(z)} = a_0 + \dots\ a_0 \neq 0$. We write $h(z) = \rho z + zg(z)$ with $g(0) = 0$. Then (1.3) becomes equivalent to

$$\rho kz + kzg(kz) = (b(z)\rho z + b(z)zg(z))(\rho + g(z) + zg'(z)),$$

where the right hand side is

$$\rho^2 zb(z) + \rho zb(z)g(z) + \rho^2 zb(z)g'(z) + \rho zb(z)g(z) + zb(z)g(z)^2 + z^2 b(z)g(z)g'(z).$$

We obtain

$$\begin{aligned}&\rho kz + kzg(kz)\\ &\qquad = \rho^2 zb(z) + 2\rho zb(z)g(z) + zb(z)g(z)^2 + g'(z)(\rho z^2 b(z) + z^2 b(z)g(z)),\end{aligned}$$

and hence we get $k = \rho a_0$ or $a_0 = k\rho^{-1}$. Then ρ is uniquely determined by $\rho = ka_0^{-1}$. In addition we obtain

$$zg'(z)(\rho b(z) + b(z)g(z)) = \rho k + kg(kz) - \rho^2 b(z) - 2\rho b(z)g(z) - b(z)g(z)^2$$

which is equaivalent to

$$zg'(z)(1+\rho^{-1}g(z)) = kb(z)^{-1} + \rho^{-1}kb(z)^{-1}g(kz) - \rho - 2g(z) - \rho^{-1}g(z)^2.$$

The series b^{-1} is given by $b(z)^{-1} = \rho k^{-1} + \dots$ and

$$(1+\rho^{-1}g(z))^{-1} = 1 - \rho^{-1}g(z) + \rho^{-2}g(z)^2 - \rho^{-3}g(z)^3 + \dots,$$

and hence we get

(2.1)

$$\begin{aligned} zg'(z) &= kb(z)^{-1} + \rho^{-1}kb(z)^{-1}g(kz) - \rho - 2g(z) - \rho^{-1}g(z)^2 - \rho^{-1}kb(z)^{-1}g(z) \\ &\quad - \rho^{-2}kb(z)^{-1}g(z)g(kz) + g(z) + 2\rho^{-1}g(z)^2 + \rho^{-2}g(z)^3 + \dots. \end{aligned}$$

The expansion of the term $-\rho^{-1}kb(z)^{-1}g(z)$ yields

$$-\rho^{-1}kb(z)^{-1}g(z) = -g(z) + zR(g(z))$$

where R is a power series. Therefore we write

$$zg'(z) = g(kz) - 2g(z) + Q_1(g(z), g(kz), z) + Q_2(z), \tag{1.4}$$

where Q_1 does not contain any term cz^ν, $c \neq 0$, $\nu \in \mathbb{N}$, ord $Q_1(g(z), g(kz), z) \geq 2$, since Q_1 collects all terms which are different from $g(kz)$, $-2g(z)$ and cz^ν, $c \neq 0$, $\nu \in \mathbb{N}$. The series Q_2 is given as $Q_2(z) = kb(z)^{-1} - \rho$. It is easy to prove that $Q_2 = 0$ if $b(z)^{-1} = a_0$ and that ord $Q_2 =$ ord $(b(z)^{-1} - a_0)$.

3. Formal solutions of the functional-differential equation of Briot-Bouquet type

In this section we prove Theorem 1.1. Before we do this we want to study when for a given $k \in \mathbb{C}$ the equation $\nu + 2 - k^\nu = 0$ holds for some $\nu \in \mathbb{N}$. The idea to investigate the equation $\nu + 2 - k^\nu = 0$ for $|k|$ is due to J. Schwaiger. Let $k \in \mathbb{C}$ be given and let $\nu_0 \in \mathbb{N}$ such that $\nu_0 + 2 - k^{\nu_0} = 0$ holds. Then $|v_0 + 2| = |k|^{\nu_0}$ follows which is the same as $\nu_0 + 2 = |k|^{\nu_0}$. Therefore it is enough to consider only values $k > 0$ respectively $k > 1$. Then the following lemma is true.

Lemma 3.1 (Oral communication J. Schwaiger)**.** *Let $k > 1$ and $f(x) = k^x - x - 2$. Then f has exactly one zero in* $[0, \infty)$.

Proof. The first derivative of f is given by $f'(x) = k^x \ln k - 1$. Therefore we get as an extremum the point $x = -\frac{\ln(\ln k)}{\ln k}$ which is a minimum of f. On the other hand for $x < -\frac{\ln(\ln k)}{\ln k}$ the function f is strictly decreasing, for $x > -\frac{\ln(\ln k)}{\ln k}$ the function f is strictly increasing, furthermore $f(0) < 0$ and $\lim_{x\to\infty} f(x) = \infty$. Therefore f has exactly one zero. □

Next we prove Theorem 1.1.

Proof of Theorem 1.1. We now write

$$g(z) = \sum_{\nu \geq 1} \gamma_\nu z^\nu \tag{3.1}$$

and we collect the 'linear' terms $g(kz)$ and $2g(z)$ on the left hand side of (1.4). Hence we obtain the equivalent equation

$$zg'(z) + 2g(z) - g(kz) = Q_1(g(z), g(kz), z) + Q_2(z). \tag{3.2}$$

Q_1 is a power series in u, w, z, say

$$Q_1(u,w,z) = \sum_{\substack{\alpha,\beta,\delta \\ \alpha+\beta+\delta\geq 2 \\ (\alpha,\beta)\neq(0,0)}} q^{(1)}_{\alpha\beta\delta} u^\alpha w^\beta z^\delta.$$

If we substitute $g(z)$ for u, $g(kz)$ for w and use (3.1) we obtain

$$Q_1(g(z),g(kz),z) = \sum_{\nu\geq 2} P_\nu(\gamma_1,\ldots,\gamma_{\nu-1},k\gamma_1,\ldots,k^{\nu-1}\gamma_{\nu-1};q^{(1)}_{\alpha\beta\delta})z^\nu$$

where P_ν is a polynomial with non-negative coefficients in $\gamma_1,\ldots,\gamma_{\nu-1}$ and in $k\gamma_1,\ldots,k^{\nu-1}\gamma_{\nu-1}$ and in certain $q^{(1)}_{\alpha\beta\delta}$, and such that

$$P_\nu(0,\ldots,0,0\ldots,0;q^{(1)}_{\alpha\beta\delta}) = 0. \tag{3.3}$$

Furthermore we use the representation

$$Q_2(z) = \sum_{\nu\geq 1} q^{(2)}_\nu z^\nu.$$

The left hand side of (3.2) is computed by

$$zg'(z) = z\left(\sum_{\nu\geq 1}\gamma_\nu z^\nu\right)' = z\left(\sum_{\nu\geq 1}\gamma_\nu \nu z^{\nu-1}\right) = \left(\sum_{\nu\geq 1}\gamma_\nu \nu z^\nu\right),$$

and

$$2g(z) - g(kz) = 2\sum_{\nu\geq 1}\gamma_\nu z^\nu - \sum_{\nu\geq 1}\gamma_\nu k^\nu z^\nu = \sum_{\nu\geq 1}(2-k^\nu)\gamma_\nu z^\nu.$$

Therefore the left hand side of (3.2) is given by

$$zg'(z) + 2g(z) - g(kz) = \sum_{\nu\geq 1}(\nu+2-k^\nu)\gamma_\nu z^\nu$$

and so we get

$$\begin{aligned}\sum_{\nu\geq 1}(\nu+2-k^\nu)\gamma_\nu z^\nu = &\sum_{\nu\geq 1} q^{(2)}_\nu z^\nu \\ &+\sum_{\nu\geq 2} P_\nu(\gamma_1,\ldots,\gamma_{\nu-1},k\gamma_1,\ldots,k^{\nu-1}\gamma_{\nu-1};q^{(1)}_{\alpha\beta\delta})z^\nu.\end{aligned} \tag{3.4}$$

We distinguish two cases.

I) First we assume that $\nu+2-k^\nu \neq 0$ for every $\nu\in\mathbb{N}$. Note that if k is a Siegel number, then $|k|^\nu = 1$ for all $\nu\in\mathbb{N}$ and hence $\nu+2-k^\nu\neq 0$ for every $\nu\in\mathbb{N}$. Therefore if k is a Siegel number or if $|k|<1$, then k belongs to this case.

It follows from (3.4) that in this situation exactly one formal solution exists. If all $q^{(2)}_\nu = 0$, that means $b(z)^{-1} = a_0$, then we get by induction $\gamma_\nu = 0$, $\nu\geq 1$. Hence also $g = 0$ and after reversing our transformations f is given by

$$f(z) = ka_0^{-1}S(z).$$

Then for the left hand side of (1.2) we obtain

$$f(\varphi(z)) = ka_0^{-1}S(\varphi(z)) = ka_0^{-1}S(S^{-1}(kS(z))) = k^2a_0^{-1}S(z),$$

and for the right hand side

$$a_0f(z)f'(z) = a_0ka_0^{-1}S(z)S'(z) = k^2a_0^{-1}S(z)S'(z).$$

Therefore $S(z) = z$ and we obtain $f(z) = ka_0^{-1}z$.
If not all $q_\nu^{(2)} = 0$, then there exists a smallest $\tilde{\nu} \geq 1$ such that $q_{\tilde{\nu}}^{(2)} \neq 0$. As a consequence of an earlier remark we have $\tilde{\nu} = \text{ord}\,(b(z)^{-1} - a_0)$. Then $\gamma_1 = \ldots = \gamma_{\tilde{\nu}-1} = 0$ while

$$\gamma_{\tilde{\nu}} = \frac{1}{\tilde{\nu} + 2 - k^{\tilde{\nu}}} q_{\tilde{\nu}}^{(2)}$$

because $P_{\tilde{\nu}}(\gamma_1, \ldots, \gamma_{\tilde{\nu}-1}, k\gamma_1, \ldots, k^{\tilde{\nu}-1}\gamma_{\tilde{\nu}-1}; q_{\alpha\beta\delta}^{(1)}) = 0$, since $\gamma_1, \ldots, \gamma_{\tilde{\nu}-1}$ are zero, see formula (3.3). Inductively we compute γ_ν, $\nu \geq \tilde{\nu}$, and so we get a solution of (1.4).
II) Now we assume that there exists a $\nu_0 \in \mathbb{N}$ such that $\nu_0 + 2 - k^{\nu_0} = 0$. Then by Lemma 3.1 but also in [1] it is shown that $\nu + 2 - k^\nu = 0$ can occur for at most one ν. It follows from (3.4) that the coefficients γ_ν, $1 \leq \nu \leq \nu_0 - 1$ are recursively, uniquely determined as

$$\gamma_\nu = \frac{1}{\nu + 2 - k^\nu}(q_{\tilde{\nu}}^{(2)} + P_\nu(\gamma_1, \ldots, \gamma_{\nu-1}, k\gamma_1, \ldots, k^{\nu-1}\gamma_{\nu-1}; q_{\alpha\beta\delta}^{(1)})).$$

For $\nu = \nu_0$ we obtain from (3.4)

$$0 = q_{\tilde{\nu}_0}^{(2)} + P_{\nu_0}(\gamma_1, \ldots, \gamma_{\nu_0-1}, k\gamma_1, \ldots, k^{\nu_0-1}\gamma_{\nu_0-1}; q_{\alpha\beta\delta}^{(1)}). \tag{3.5}$$

This relation can be written as a special relation between certain coefficients of Q_1 and Q_2 and therefore as polynomial relation between the coefficients of b^{-1} and hence of a itself. The polynomial relation (3.5) is sufficient and necessary for the existence of a formal solution g. If the polynomial relation (3.5) is satisfied we can choose the coefficient $\gamma_{\nu_0} \in \mathbb{C}$ arbitrarily. Then again the coefficients γ_ν with $\nu > \nu_0$ are uniquely determined by (3.4). Finally if we consider the situation when $b(z)^{-1} = a_0$ and hence $Q_2(z) = 0$, then it follows from (3.4) that $\gamma_1 = \ldots = \gamma_{\nu_0-1} = 0$. Therefore the solution g is not zero if and only if γ_{ν_0} is choosen such that $\gamma_{\nu_0} \neq 0$.

After computing the non-trivial formal solution g of (1.4) we can reverse our calculations in both cases and so we get as a formal solution of (1.2)

$$f(z) = \rho S(z) + S(z)g(S(z)). \tag{3.6}$$

□

4. Proof of the convergence of the formal solutions

Next we have to show that the formal solutions in (3.6) are holomorphic. Therefore it is sufficient to show that g is local analytic, then the solutions in (3.6) are also local analytic (see [2], Chapter 1.2).

Proof of Theorem 1.3. Again we study the cases I) and II) from Section 3.
I) For every $\nu \in \mathbb{N}$ we have $\nu + 2 - k^\nu \neq 0$. Let $|k| > 1$, then we get

$$|\nu + 2 - k^\nu| = \left| k^\nu \frac{v+2}{k^\nu} - k^\nu \right| = |k|^\nu \left| \frac{v+2}{k^\nu} - 1 \right|.$$

The expression $\left|\frac{v+2}{k^\nu} - 1\right|$ converges to 1 for $\nu \to \infty$ and hence there exists a $C > 0$ such that

$$|\nu + 2 - k^\nu| \geq C\,|k|^\nu \tag{4.1}$$

for $\nu \in \mathbb{N}$. For the convergence proof we use the method of majorants and therefore we introduce the following problem, which is a majorant problem to equation (3.2).

Let $\hat{Q}_1(u, v, w)$ be a convergent majorant of Q_1, ord $\hat{Q}_1 \geq 2$, which has no terms in w^ν for $\nu \in \mathbb{N}$, and let $\hat{Q}_2(z)$ be a convergent majorant of Q_2 with ord $\hat{Q}_2 \geq 1$. Then the equation

$$C\hat{g}(|k|\, z) = \hat{Q}_2(z) + \hat{Q}_1(\hat{g}(z), \hat{g}(|k|\, z), z) \tag{4.2}$$

for the series $\hat{g}(z) = \hat{\gamma}_1 z + \ldots$ leads to a system of equations which is analogous to (3.4), namely

$$C\,|k|^\nu\, \hat{\gamma}_\nu = \hat{q}_\nu^{(2)} + P_\nu(\hat{\gamma}_1, \ldots, \hat{\gamma}_{\nu-1}, |k|\, \hat{\gamma}_1, \ldots, |k|^{\nu-1}\, \hat{\gamma}_{\nu-1}; \hat{q}^{(1)}_{\alpha\beta\delta}),$$

for $\nu \geq 1$, with the corresponding coefficient $\hat{q}_\nu^{(2)}$ of $\hat{Q}_2(z)$ and with $\hat{q}^{(1)}_{\alpha\beta\delta}$ of the series $\hat{Q}_1(\hat{g}(z), \hat{g}(|k|\, z), z)$. The polynomials P_ν are the same as in (3.4). By induction we see that

$$\hat{\gamma}_\nu \geq 0 \text{ and } |\gamma_\nu| \leq \hat{\gamma}_\nu,\ \nu \geq 1.$$

We now consider a further majorant problem, namely

$$C\hat{\hat{g}}(|k|\, z) = \hat{Q}_2(z) + \hat{Q}_1(\hat{\hat{g}}(|k|\, z), \hat{\hat{g}}(|k|\, z), z). \tag{4.3}$$

Hence we obtain the following system of equations for the coefficients $\hat{\hat{\gamma}}_\nu$ of $\hat{\hat{g}}$

$$C\,|k|^\nu\, \hat{\hat{\gamma}}_\nu = \hat{q}_\nu^{(2)} + P_\nu(\hat{\hat{\gamma}}_1, \ldots, \hat{\hat{\gamma}}_{\nu-1}, |k|\, \hat{\hat{\gamma}}_1, \ldots, |k|^{\nu-1}\, \hat{\hat{\gamma}}_{\nu-1}; \hat{q}^{(1)}_{\alpha\beta\delta}).$$

Because $|k| > 1$ we obtain by induction

$$\hat{\hat{\gamma}}_\nu \geq \hat{\gamma}_\nu,\ \nu \geq 1.$$

The problem (4.3) is an implicit function problem (see [4], p. 63) for $\hat{\hat{g}}$ and hence $\hat{\hat{g}}$ is analytic in $z = 0$. Therefore also g is a convergent power series.
Let k be a Siegel number, then $|k| = 1$ or let $|k| < 1$. In both situations we get

$$|\nu + 2 - k^\nu| \geq \nu + 2 - |k|^\nu \geq \nu + 1$$

and hence there exists a $C > 0$ such that $|\nu + 2 - k^\nu| > C$. For the Siegel number k again we construct the majorant problem

$$C\hat{g}(|k|\, z) = \hat{Q}_2(z) + \hat{Q}_1(\hat{g}(z), \hat{g}(|k|\, z), z) \tag{4.4}$$

for $\hat{g}(z) = \hat{\gamma}_1 z + \ldots$, which, as above, leads to

$$C\hat{\gamma}_\nu = \hat{q}_\nu^{(2)} + P_\nu(\hat{\gamma}_1, \ldots, \hat{\gamma}_{\nu-1}, \hat{\gamma}_1, \ldots, \hat{\gamma}_{\nu-1}; \hat{q}^{(1)}_{\alpha\beta\delta}),$$

for $\nu \geq 1$. Problem (4.4) is the same as the problem

$$C\hat{g}(z) = \hat{Q}_2(z) + \hat{Q}_1(\hat{g}(z), \hat{g}(z), z) \tag{4.5}$$

because $|k| = 1$, and hence we get a local analytic solution $\hat{g}$ of the implicit function problem (4.5). Therefore also g is local analytic. If $|k| < 1$ then we consider the majorant problem

$$C\hat{g}(|k|\, z) = \hat{Q}_2(z) + \hat{Q}_1(\hat{g}(z), \hat{g}(|k|\, z), z) \tag{4.6}$$

and because $|k| < 1$ we have the majorant problem

$$C\hat{\hat{g}}(z) = \hat{Q}_2(z) + \hat{Q}_1(\hat{\hat{g}}(z), \hat{\hat{g}}(z), z), \tag{4.7}$$

and hence there exists a local analytic solution $\hat{\hat{g}}$ and hence also g is local analytic.
II) Let $\nu_0 + 2 - k^{\nu_0} = 0$ for one $\nu_0 \in \mathbb{N}$. Similar to case I) there exists a $C > 0$ such that

$$|\nu + 2 - k^\nu| \geq C\,|k|^\nu \tag{4.8}$$

holds for all $\nu \neq \nu_0$. For $\hat{g}(z) = \hat{\gamma}_1 z + \dots$ we construct the majorant problem

$$C\hat{g}(|k|\,z) = Dz^{\nu_0} + \hat{Q}_2(z) + \hat{Q}_1(\hat{g}(z), \hat{g}(|k|\,z), z) \tag{4.9}$$

where $D > 0$ is chosen appropriately, that means D has to be choosen sufficiently large. For $\nu = 1, \dots \nu_0 - 1$ we obtain

$$|\gamma_\nu| \leq \hat{\gamma}_\nu.$$

For $\nu = \nu_0$ we have

$$C\,|k|^\nu\,\hat{\gamma}_{\nu_0} = D + \hat{q}_\nu^{(2)} + P_\nu(\hat{\gamma}_1, \dots, \hat{\gamma}_{\nu-1}, |k|\,\hat{\gamma}_1, \dots, |k|^{\nu-1}\,\hat{\gamma}_{\nu-1}; \hat{q}^{(1)}_{\alpha\beta\delta}).$$

Then $D > 0$ has to be large enough such that

$$|\gamma_{\nu_0}| \leq \hat{\gamma}_{\nu_0}$$

holds, where γ_{ν_0} is the freely selectable coefficient of z^{ν_0} in g. In the next step we consider

$$C\hat{\hat{g}}(|k|\,z) = Dz^{\nu_0} + \hat{Q}_2(z) + \hat{Q}_1(\hat{\hat{g}}(|k|\,z), \hat{\hat{g}}(|k|\,z), z), \tag{4.10}$$

which is a majorant problem of (4.9). Again we can determine the function $\hat{\hat{g}}$ by the implicit function theorem and hence $\hat{\hat{g}}$ is analytic in $z = 0$. Then it follows that g is a local analytic function.
So it results from our two cases that all formal solutions f of equation (1.2) are local analytic. □

5. Alternative transformations

In this section we want to discuss two different transformations of the functional-differential equation $f(\varphi(z)) = a(z)f(z)f'(z)$ to a functional-differential equation of Briot-Bouquet type.

5.1. **Transformation 1.** We can start our considerations with

$$h(kz) = b(z)h(z)h'(z) \tag{1.3}$$

where b is given by $b(z) = \frac{a(S^{-1}(z))}{(S^{-1})'(z)} = a_0 + \dots$, $a_0 \neq 0$. We write $h(z) = \rho z\,\mathrm{e}^{g(z)}$. Then we have

$$h'(z) = \rho \mathrm{e}^{g(z)} + \rho z \mathrm{e}^{g(z)} g'(z),$$

and so (1.3) becomes equivalent to

$$\rho k z \mathrm{e}^{g(kz)} = b(z)\rho z \mathrm{e}^{g(z)}(\rho \mathrm{e}^{g(z)} + \rho z \mathrm{e}^{g(z)} g'(z)),$$

which is the same as

$$k\mathrm{e}^{g(kz)} = b(z)\rho \mathrm{e}^{2g(z)}(1 + zg'(z)). \tag{5.1}$$

From (5.1) we obtain

$$\rho^{-1} k b(z)^{-1} \mathrm{e}^{g(kz) - 2g(z)} = 1 + zg'(z),$$

and hence we get $ka_0^{-1}\rho^{-1} = 1$ or $\rho = ka_0^{-1}$. By substituting $g(kz) - 2g(z)$ with ord $(g(kz) - 2g(z)) > 0$ into the exponential series we obtain

$$(1 + \rho^{-1} k a_1 z + \dots)(1 + g(kz) - 2g(z) + \dots) = 1 + zg'(z),$$

which is equaivalent to

$$1 + g(kz) - 2g(z) + \dots = 1 + zg'(z).$$

Therefore we write

$$zg'(z) = g(kz) - 2g(z) + F_1(g(z), g(kz), z) + F_2(z), \tag{5.2}$$

where again F_1 does not contain a term Az^m, $A \neq 0$, $m \in \mathbb{N}$, and also again for F_1 we have ord $F_1(g(z), g(kz), z) \geq 2$. The series F_2 is given by $F_2(z) = kb(z)^{-1} - \rho$. Equation (5.2) is again a functional-differential equation of Briot-Bouquet type. After computing the solutions g of (5.2) we can reverse our transformations and so the solutions f of

$$f(\varphi(z)) = a(z)f(z)f'(z) \tag{1.2}$$

are given by

$$f(z) = ka_0^{-1}S(z)\mathrm{e}^{g(S(z))}.$$

5.2. **Transformation 2.** For this second transformation we also start with the equation

$$h(kz) = b(z)h(z)h'(z) \tag{1.3}$$

where b is given by $b(z) = \frac{a(S^{-1}(z))}{(S^{-1})'(z)} = a_0 + \ldots$, $a_0 \neq 0$. Then we observe that $\left(h(z)^2\right)' = 2h(z)h(z)'$ and so (1.3) becomes equivalent to

$$h(kz) = \frac{b(z)}{2}\left(h(z)^2\right)'.$$

Now we write $h(z) = \rho z(1 + g(z))$ where $g(0) = 0$ and therefore we have

$$h(z)^2 = \rho^2 z^2 (1 + g(z))^2 = \rho^2 z^2 (1 + \tilde{g}(z))$$

where $\tilde{g}(0) = 0$. Hence, by the binomial series we get conversely

$$h(z) = \rho z(1 + \tilde{g}(z))^{\frac{1}{2}}.$$

Then (1.3) is equivalent to

$$\rho kz(1 + \tilde{g}(kz))^{\frac{1}{2}} = \frac{b(z)}{2}\left(2\rho^2 z(1 + \tilde{g}(z)) + \rho^2 z^2 \tilde{g}'(z)\right)$$

or to

$$k(1 + \tilde{g}(kz))^{\frac{1}{2}} = \frac{b(z)}{2}\left(2\rho(1 + \tilde{g}(z)) + \rho z\tilde{g}'(z)\right)$$

after we cancel ρ and z. This is the same as

$$2b(z)^{-1}k(1 + \tilde{g}(kz))^{\frac{1}{2}} = 2\rho(1 + \tilde{g}(z)) + \rho z\tilde{g}'(z),$$

and so we obtain

$$z\tilde{g}'(z) = \rho^{-1}2b(z)^{-1}k(1 + \tilde{g}(kz))^{\frac{1}{2}} - 2 - 2\tilde{g}(z). \tag{5.3}$$

If we compare the absolute terms in (5.3) we get $0 = \rho^{-1}2ka_0^{-1} - 2$ and hence $0 = 2(\rho^{-1}ka_0^{-1} - 1)$. Again we obtain

$$\rho = ka_0^{-1}.$$

By using the binomial series we get

$$(1 + \tilde{g}(kz))^{\frac{1}{2}} = \sum_{\nu=0}^{\infty}\binom{\frac{1}{2}}{\nu}\tilde{g}(kz)^{\nu} = 1 + \frac{1}{2}\tilde{g}(kz) + \binom{\frac{1}{2}}{2}\tilde{g}(kz)^2 + \ldots.$$

Hence equation (5.3) can be computed as follows

$$\begin{aligned} z\tilde{g}'(z) &= 2\rho^{-1}k(a_0^{-1} + b_1 z + \dots)\left(1 + \frac{1}{2}\tilde{g}(kz) + \dots\right) - 2 - 2\tilde{g}(z) \\ &= (2 + 2b_1\rho^{-1}kz + \dots)\left(1 + \frac{1}{2}\tilde{g}(kz) + \dots\right) - 2 - 2\tilde{g}(z) \\ &= 2 + 2b_1\rho^{-1}kz + \dots + \tilde{g}(kz) + b_1\rho^{-1}kz\tilde{g}(kz) + \dots - 2 - 2\tilde{g}(z). \end{aligned}$$

In this equation 2 cancels and so again we obtain a functional-differential equation of Briot-Bouquet type, namely

$$z\tilde{g}'(z) = -2\tilde{g}(z) + \tilde{g}(kz) + G_1(\tilde{g}(z), \tilde{g}(kz), z) + G_2(z), \tag{5.4}$$

where G_1 does not contain any absolute term and ord $G_1 \geq 2$, ord $G_2 \geq 1$. After solving equation (5.4) we reverse our transfomations and we get

$$f(z) = ka_0^{-1}S(z)(1 + \tilde{g}(S(z)))^{\frac{1}{2}}$$

as solutions of the functional differential equation

$$f(\varphi(z)) = a(z)f(z)f'(z). \tag{1.2}$$

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