GENERALIZED DHOMBRES EQUATIONS IN THE COMPLEX DOMAIN
A SURVEY

LUDWIG REICH AND JÖRG TOMASCHEK

Abstract. In this survey paper we present the Main Theorems related to the generalized Dhombres equation in the complex domain. We discuss the local as well as the global theory. This includes formal, local analytic, entire and meromorphic solutions.

Contents

Introduction

The generalized Dhombres functional equation in the complex domain is given by

(GDh,φ) $$f(zf(z)) = \varphi(f(z)),$$

where the function φ is known and f is unknown. (GDh,φ) belongs to the class of iterative functional equations (see [7] for a survey on these equations). The main difficulty in solving such equations comes from the fact that the unknown function f or an expression containing f is substituted into f. For a complex variable z and analytic functions φ and f, this equation was first considered by L. Reich, J. Smítal and M. Štefánková in [14] in the year 2005. The origin of this equation dates back

J. Tomaschek is supported by the National Research Fund, Luxembourg (AFR 3979497), and cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND).

to 1975. As it is indicated in the equation's name, J. Dhombres was the one who introduced this equation in the year 1975 in [2], and four years later in [3]. In the previous years the generalized Dhombres equation was also investigated for a real variable. Here we mention P. Kahlig and J. Smítal and we refer the reader to the article [12] and the references given there.

In this survey paper we want to describe, as the title displays, the so called local problem as well as the global problem for a complex variable. Motivated by the classical theory of ordinary differential equations in the complex domain (see [6]) we start by investigating the local problem from which we try to obtain some conclusions for the global problem. For the local problem we mainly use the method of formal power series, but also representations involving infinite products etc. For the global problem clearly also other arguments are necessary. The remaining part of this introduction is used to give a short overview on both problems.

The local problem is given as follows.
Let D be a region in $\mathbb{C}$, $w_0 \in D$, and let $\varphi\colon D \to \mathbb{C}$ be holomorphic. Let $z_0 \in \mathbb{C}$. By a local analytic solution of

$$\text{(GDh,}\varphi\text{)} \qquad f(zf(z)) = \varphi(f(z))$$

we mean a function f, holomorphic in a neighbourhood $\{z\colon |z - z_0| < \epsilon(f)\}$ of z_0, such that $f(z_0) = w_0$ and such that (GDh,φ) holds for all z with $|z - z_0| < \epsilon(f)$. $\epsilon(f)$ may depend on the particular solution f.

The global problem is decribed by the following setting.
Let D be a region in $\mathbb{C}$, $w_0 \in D$ and let $\varphi\colon D \to \mathbb{C}$ be holomorphic. Assume that G is a region in $\mathbb{C}$, $z_0 \in G$. Does there exist a holomorphic function $f\colon G \to \mathbb{C}$ such that $f(z_0) = w_0$ and such that f fulfills (GDh,φ) for all $z \in G$? Then we call f a global solution of (GDh,φ). If f is a global solution the restriction

$$f|_{B_\epsilon(z_0)}$$

is a local analytic solution of (GDh,φ) for every sufficiently small ϵ.

It is difficult to find regions G and functions φ for which the global problem has non-constant solutions.

The formal solutions.
We use the following method, which consists in parts of some well sophisticated transformations, to compute formal solutions. Let f be a local analytic solution of (GDh,φ) with $f(z_0) = w_0$. Then we can write $f(z) = w_0 + g(z)$, where g is holomorphic at z_0 and $g(z_0) = 0$. Then φ is given by $\varphi(y) = \varphi(w_0) + \tilde{\varphi}(y - w_0)$, $\tilde{\varphi}(0) = 0$, $z = z_0 + \zeta$. Introducing $h(\zeta) = g(z)$, $h(0) = 0$ gives us

$$\text{(GDh,}\varphi\text{)} \;\Leftrightarrow\; h(z_0w_0 - z_0 + w_0\zeta + z_0h(\zeta) + \zeta h(\zeta)) = \varphi(w_0) - w_0 + \tilde{\varphi}(h(\zeta))$$

in a neighbourhood of 0. We may there consider the Taylor expansions of h and $\tilde{\varphi}$ at 0. Only if $z_0w_0 - z_0 = 0$, we get a relation which makes sense as a relation for formal power series, since then substitution of $w_0\zeta + z_0h(\zeta) + \zeta h(\zeta)$ into h is defined. We have $\varphi(w_0) = w_0$ and

$$0 = z_0w_0 - z_0 \;\Leftrightarrow\; z_0 = 0, w_0 \in \mathbb{C} \;\underline{\text{or}}\; z_0 \in \mathbb{C} \setminus \{0\}, w_0 = 1.$$

Also the situations $f(\infty)=\infty$ and $f(\infty)=w_0\in\mathbb{C}\setminus\{0\}$ are accessible to formal power series methods, see [25].

Transformations of (GDh,φ) in the case $f(0)=w_0\in\mathbb{C}$.
Necessary conditions for the existence of non-constant analytic solutions are:

(1) If $f(0)=0$, $f\neq 0$, then there exists a $k\in\mathbb{N}$, $c_k\in\mathbb{C}\setminus\{0\}$ such that

$$f(z)=c_kz^k+\ldots$$

and as consequence of (GDh,φ) $\qquad \varphi(y)=y^{k+1}+d_{k+1}y^{k+2}+\ldots$

in a neighbourhood of 0. This was the first transformation of such a type. It was introduced in [14].

(2) If $f(0)=w_0\in\mathbb{C}\setminus\{0\}$, $f\neq w_0$, then there exists a $k\in\mathbb{N}$, g holomorphic at 0, such that $f(z)=w_0+g(z)$, ord $g=k$ and

$$\tilde{\varphi}(y)\equiv w_0^ky \pmod{y^2},\ \varphi(w_0)=w_0$$
$$(\mathrm{GDh},\varphi)\ \Leftrightarrow\ g(z(w_0+g(z))=\tilde{\varphi}(g(z)).$$

This was published in [16].

Later on transformations for the cases where $f(z_0)=1$ for $z_0\neq 0$ and for solutions f of (GDh,φ) in a neighbourhood of infinity were used. They are given in [22] and in [25] for the "infinity" situation.

At the end of this introduction we want to give a short description of the definitions which are used in this article. By $\mathbb{E}$ we denote the set of the complex roots of one, $\mathbb{C}[\![z]\!]=\{F\colon F(z)=\beta_0+\beta_1z+\beta_2z^2+\ldots\}$ with usual addition, multiplication and substitution is the ring of formal power series. The order of a non trivial series $F\in\mathbb{C}[\![z]\!]$ is defined by

$$\mathrm{ord}\ F:=\min\{\nu\in\mathbb{N}\colon \beta_\nu\neq 0\}$$

and one sets ∞ for the order of the trivial series. The group Γ_1 together with substitution is defined by $\Gamma_1=\{F\colon F\in\mathbb{C}[\![z]\!]$ and $F(z)\equiv z \pmod{z^2}\}$ and it is a subgroup of $\Gamma=\{F\colon F\in\mathbb{C}[\![z]\!]$ and $F(z)=az+\ldots\}$. For a systematic introduction of formal power series see [1] or [5].

1. Formal and local analytic solutions

In this section we discuss the local problem. For all known initial values of a solution f of (GDh,φ) we present formal but also local analytic results.

1.1. $f(0)=0$. **The Böttcher case.** The results of this section concerning the generalized Dhombres equation can be found in [14], in [11] the theory on Böttcher functional equations is provided. We start with an existence theorem. One of the most beautiful outcomes in the whole theory is the existence of a so called generator function $\tilde{g}_0$, which allows us in many cases to write all solutions of (GDh,φ) in a convenient way, as we will see in the first theorem. Let us also mention here, that various generators will accompany us through this article. For all solvable equations (GDh,φ) there exists a set of solutions of the form

$$\{\tilde{g}_0(c_kz^k)\colon c_k\in\mathbb{C}\},\ \text{where}\ \tilde{g}_0\neq 0,$$

but in some cases (Theorem 1.20) the set of all solutions is strictly larger. Nevertheless, also in this situation the description of the general solution is based on the subset of solutions given by a "subgenerator" $\tilde{g}_0$.

Theorem 1.1 (Existence and structure theorem, formal part)**.** *Let $\varphi(y) \in \mathbb{C}[\![y]\!]$, $f(z) \in \mathbb{C}[\![z]\!]$, ord $f = k \geq 1$, then the following holds.*

(1) *The generalized Dhombres equation*

$$f(zf(z)) = \varphi(f(z)) \tag{GDh,φ}$$

has a non constant solution $f \in \mathbb{C}[\![z]\!]$ if and only if

$$\varphi(y) = y^{k+1} + d_{k+2}y^{k+2} + \dots \tag{1.1}$$

holds.

(2) *If $\varphi(y)$ has the form* (1.1) *(for a $k \in \mathbb{N}$), then the set $\mathcal{L}_\varphi$ of all formal solutions of* (GDh,φ) *with $f(0) = 0$ can be described as follows:*
There exists a $\tilde{g}_0 \in \mathbb{C}[\![y]\!]$, $\tilde{g}_0(y) = y + \dots$, uniquely determined by φ, such that

$$\mathcal{L}_\varphi = \left\{ f\colon f \in \mathbb{C}[\![z]\!], f(z) = \tilde{g}_0(c_k z^k), c_k \in \mathbb{C} \right\}. \tag{1.2}$$

$\tilde{g}_0$ is called the generator of (GDh,φ)*.*

(3) *If $\tilde{g}_0$ is the generator of* (GDh,φ)*, then*

$$\varphi(y) = \tilde{g}_0(y^k \tilde{g}_0^{-1}(y)), \tag{1.3}$$

and $h_0 = \tilde{g}_0^{-1}$ is the unique solution of

$$\begin{cases} y^k h_0(y) = h_0(\varphi(y)) \\ \quad h_0(y) = y + \dots \end{cases} \tag{1.4}$$

Naturally after the complete description of formal solutions, we are interested in local analytic ones. The following theorem is devoted to this topic.

Theorem 1.2 (Local analytic solutions)**.** *Let $\varphi(y) \in \mathbb{C}[\![y]\!]$, $f(z) \in \mathbb{C}[\![z]\!]$, ord $f = k \geq 1$ and assume that $\varphi(y) = y^{k+1} + d_{k+2}y^{k+2} + \dots$ is holomorphic at $y = 0$. Then the following holds true.*

(1) *All series f of the set* (1.2) *are convergent, hence yield local analytic solutions of* (GDh,φ)*.*
(2) *In particular, the generator $\tilde{g}_0$ of* (GDh,φ) *is convergent. If* (GDh,φ) *has one non-constant local analytic solution f with $f(0) = 0$, then all such formal solutions are convergent. (Clearly* (1.3) *and* (1.4) *hold for local analytic functions).*

Sketch of the proof of Theorem 1.1 and Theorem 1.2. Let φ be given by $\varphi(y) = y^{k+1} + d_{k+2}y^{k+2} + \dots$ $(k \geq 1)$. Then there exists exactly one function B_φ, $B_\varphi(y) = y + \dots$ such that

$$\varphi(y) = B_\varphi^{-1}(B_\varphi(y)^{k+1}), \tag{1.5}$$

the function B_φ is called Böttcher function of φ. We define $g := B_\varphi \circ f$ (ord $g = k$). By (1.5) and (GDh,φ) we obtain

$$(\mathrm{GDh},\varphi) \;\Leftrightarrow\; g(zB_\varphi^{-1}(g(z))) = g(z)^{k+1}.$$

Write, without loss of generality $g(z) = T(z)^k$, ord $T = 1$, $U := T^{-1}$, then

$$\text{(GDh,}\varphi) \Leftrightarrow \begin{cases} B_\varphi^{-1}(z^k)U(z) = U(z^{k+1}) \\ \text{ord } U = 1. \end{cases} \tag{1.6}$$

(1.6) is a linear functional equation for U, but also a generalized Böttcher equation. (From the theory of Böttcher functional equations it follows, that for $u \in \mathbb{C} \setminus \{0\}$ there exists a unique solution $U(u) = uz + \ldots$. If φ is convergent, then each U is, too.)
In order to get the structure of the set of solutions $\mathcal{L}_\varphi$ we proceed as follows. We write $U(z) = uzV(z)$, $V(z) = 1 + \ldots$, $u \neq 0$. Then

$$\text{(GDh,}\varphi) \Leftrightarrow \frac{B_\varphi^{-1}(z^k)}{z^k} V(z) = V(z^{k+1}). \tag{1.7}$$

We note that $\frac{B_\varphi^{-1}(z^k)}{z^k} = 1 + \ldots \in \mathbb{C}[\![z^k]\!]$. Now we take the formal logarithm Ln on both sides of (1.7), $\operatorname{Ln} \frac{B_\varphi^{-1}(z^k)}{z^k} =: \tilde{C}(z^k)$, $\operatorname{Ln} V =: X$ (ord $X \geq 1$, ord $\tilde{C} \geq 1$). Hence we obtain

$$\tilde{C}(z^k) + X(z) = X(z^{k+1}). \tag{1.8}$$

Since the homogeneous equation to (1.8) has only the solution 0, the substitution $z \to e^{\frac{2\pi i}{k}} z$ shows that $X(e^{\frac{2\pi i}{k}} z) = X(z)$, hence $X(z) = \tilde{X}(z^k)$ for a series $\tilde{X}$, ord $\tilde{X} \geq 1$. Therefore we have

$$\text{(GDh,}\varphi) \Leftrightarrow \tilde{C}(y) + \tilde{X}(y) = \tilde{X}(y^{k+1}). \tag{1.9}$$

Equation (1.9) has the unique solution

$$\tilde{X}(y) = -\sum_{\nu=0}^{\infty} \tilde{C}(y^{(k+1)^\nu}). \tag{1.10}$$

This converges, in the case of formal solutions, in the order topology (the convergence is very fast). If φ is convergent, then (1.10) converges uniformly in each compact subset of a certain neighbourhood of 0, and hence (1.10) is analytic. By transforming back we get

$$X(z) = -\sum_{\nu=0}^{\infty} \tilde{C}(z^{k(k+1)^\nu})$$

and

$$V(z) = \exp\left(-\sum_{\nu=0}^{\infty} \tilde{C}(z^{k(k+1)^\nu})\right).$$

Furthermore we obtain

$$U(z) = uz \exp\left(-\sum_{\nu=0}^{\infty} \tilde{C}(z^{k(k+1)^\nu})\right), \qquad u \in \mathbb{C} \setminus \{0\},$$

where $zV(z) \in z\mathbb{C}[\![z^k]\!]$, which does not depend on u. We write $U(z) = u\tilde{U}(z) = (L_u \circ \tilde{U})(z)$, where $\tilde{U}$ does not depend on u and $\tilde{U}(z) \in z\mathbb{C}[\![z^k]\!] \cap \Gamma_1$. One can show that

$$T(z) = U^{-1}(z) = \tilde{U}^{-1}(L_{u^{-1}})(z) \in \frac{z}{u}\mathbb{C}\left[\!\!\left[\left(\frac{z}{u}\right)^k\right]\!\!\right],$$

hence $T(z)^k = g(z) \in \mathbb{C}[\![z^k]\!]$, and $g(z) = \tilde{\tilde{g}}_0\left(\left(\frac{z}{u}\right)^k\right)$. With $f = B_\varphi^{-1} \circ g$, $\frac{1}{u^k} := c_k$ we get the representation $f(z) = \tilde{g}_0(c_k z^k)$, $c_k \in \mathbb{C}$. where $\tilde{g}_0(y) = y + \dots$ is the <u>same</u> for <u>all</u> $c_k \in \mathbb{C} \setminus \{0\}$. □

Remark 1.3 (to Theorem 1.1 and Theorem 1.2). Under the hypothesis of these theorems we have:
To each $c_k \in \mathbb{C}$ there exists exactly one formal (resp. local analytic) solution f with

$$f(z) = c_k z^k + \dots (= \tilde{g}_0(c_k z^k)).$$

1.1.1. *The converse problem.* Let f with $f(0) = 0$, ord $f = k \in \mathbb{N}$ be given. When is this f a formal (resp. local analytic) solution of an equation (GDh,φ)? The answer gives the following theorem.

Theorem 1.4. *Let $f_0 \in \mathbb{C}[\![z]\!]$, ord $f = k \in \mathbb{N}$. Then the following holds true.*

(1) *There exists $\varphi \in \mathbb{C}[\![y]\!]$ such that f_0 is a solution of* (GDh,φ) *if and only if there exists $g \in \mathbb{C}[\![z]\!]$, with ord $g = 1$ such that*

$$f_0(z) = g(z^k). \tag{1.11}$$

(2) *φ is uniquely determined by f_0, and the generator of* (GDh,φ) *is given by writing*

$$f_0(z) = \tilde{g}_0(c_k^{(0)} z^k) \tag{1.12}$$

if $f_0(z) = c_k^{(0)} z^k + \dots$, $c_k^{(0)} \neq 0$.

Remember equation (1.3), *namely $\varphi(y) = \tilde{g}_0(y^k \tilde{g}_0^{-1}(y))$.*
φ and $\tilde{g}_0$ are convergent if and only if f_0, satisfying (1.12), *is convergent.*

The solution of the converse problem for local analytic f leads to the important

Remark 1.5. Let $\tilde{g}_0$ be a series with $\tilde{g}_0(y) = y + \dots$, which has a radius of convergence $\rho(\tilde{g}_0)$ satisfying

$$0 < \rho(\tilde{g}_0) < +\infty.$$

Take $\varphi(y) = \tilde{g}_0(y^k \tilde{g}_0^{-1}(y))$ which is holomorphic at 0 and consider (GDh,φ) with the generator $\tilde{g}_0$. Then

$$\lim_{|c_k| \to \infty} \rho(\tilde{g}_0(c_k z^k)) = 0,$$

hence the local analytic solutions of (GDh,φ) do not have a common neighbourhood of 0 where they are defined and satisfy (GDh,φ).

1.1.2. (GDh,φ) *and Briot-Bouquet differential equations and iteration groups.* In [14] also the connection between the generalized Dhombres equation and Briot-Bouquet differential equations as well as iteration groups was established. The following results deal with the set of solutions of the respective connections. We start by introducing Briot-Bouquet differential equations, as a reference we mention [6].

A Briot-Bouquet differential equation is a differential equation of the form

$$\begin{cases} z\dfrac{\mathrm{d}\,w}{\mathrm{d}\,z} = F(z, w(z)) \\ w(0) = 0 \end{cases}$$

where F is holomorphic at $(0,0)$, $F(0,0)=0$. Let $k\geq 1$ and $\varphi(y)=y^{k+1}+d_{k+2}y^{k+2}+\ldots$ holomorphic at 0. Then we have the following characterization of the connection between generalized Dhombres equations and Briot-Bouquet differential equations.

Theorem 1.6. *The set of all local analytic solutions $\mathcal{L}_\varphi$ of* (GDh,φ) *(with the φ given above) is the set of all local analytic solutions w with $w(0)=0$ of the Briot-Bouquet differential equation*

$$z\frac{\mathrm{d}\,w}{\mathrm{d}\,z}=kN_0(w(z))$$

where

$$N_0(y)=\tilde{g}_0^{-1}(y)\cdot\tilde{g}_0'(\tilde{g}_0^{-1}(y)) \tag{1.13}$$

with the generator $\tilde{g}_0$ of (GDh,φ).

The theory of Briot-Bouquet differential equations gives a converse result to Theorem 1.6.

Theorem 1.7. *Let $N_0(y)=y+\ldots$ be holomorphic at $y=0$, $k\in\mathbb{N}$. Then there exists a unique convergent series $\varphi(y)(=y^{k+1}+d_{k+2}y^{k+2}+\ldots)$ such that the set $\mathcal{L}_\varphi$ of all local analytic solutions f, $f(0)=0$ of* (GDh,φ) *is the same as the set of local analytic solutions w of*

$$z\frac{\mathrm{d}\,w}{\mathrm{d}\,z}=kN_0(w(z)),\qquad w(0)=0.$$

The generator $\tilde{g}_0$ of this (GDh,φ) *is the unique solution of*

$$y\tilde{g}_0'(y)=N_0(\tilde{g}_0(y)),\qquad \tilde{g}_0(y)=y+\ldots. \tag{1.14}$$

By a local change of coordinates it is possible to transform (1.14) to an Aczél-Jabotinsky differential equation. Therefore let $\tilde{g}_0$ with $\tilde{g}_0(y)=y+\ldots$ be given by (1.14). Then

$$N_0(z)\cdot\phi'(z)=N_0(\phi(z)),\quad \phi(0)=0,\quad \phi\neq 0 \tag{1.15}$$

$$\Leftrightarrow$$

$$\Psi=\phi\circ\tilde{g}_0\ \text{ satisfies }\ z\Psi'(z)=N_0(\Psi(z)). \tag{1.14}$$

(Conversely, if Ψ with $\Psi(0)=0$, $\Psi\neq 0$ fulfills (1.14), then $\phi:=\Psi\circ\tilde{g}_0^{-1}$ satisfies (1.15).) Equation (1.15) is an Aczél-Jabotinsky differential equation, for the theory of this type of differential equations we refer the reader to [9] and [10]. It is well known:

The set of all local analytic solutions ϕ with $\phi(0)=0$, $\phi\neq 0$ of (1.15) forms an analytic iteration group of type I, i.e. they form a solution $(F(t,z))_{t\in\mathbb{C}}$ with

$$F(t,z)=e^tz+\sum_{\nu\geq 2}P_\nu(e^t)z^\nu\qquad (t\in\mathbb{C})$$

of the translation equation

$$F(t,F(s,z))=F(t+s,z)\qquad (t,s\in\mathbb{C})$$

which is valid in a neighbourhood of $z=0$ (depending on (t,s)), the P_ν, $\nu\geq 2$ are universal polynomials. N_0 is the "infinitesimal generator" of $(F(t,z))_{t\in\mathbb{C}}$; $N_0(z)=\frac{\partial}{\partial t}\,F(t,z)|_{t=0}$. This connection gives

Theorem 1.8. *Let $\varphi(y) = y^{k+1} + d_{k+2}y^{k+2} + \dots$ be holomorphic at $y = 0$, $k \geq 1$. Then the following holds:*
There exists a unique analytic iteration group $(F(t,z))_{t\in\mathbb{C}}$, $F(t,z) = e^t z + \dots$, such that the set of all non-zero local analytic solutions of (GDh,φ), $\mathcal{L}_\varphi \setminus \{0\}$, is given by the series

$$f^\star(t,z) = F(t, \tilde{g}_0(z^k)) \qquad (t \in \mathbb{C}) \tag{1.16}$$

where $\tilde{g}_0$ is the generator of (GDh,φ). The infinitesimal generator of $(F(t,z))_{t\in\mathbb{C}}$ is given by

$$N_0(y) = \tilde{g}_0^{-1}(y) \cdot \tilde{g}_0'(\tilde{g}_0^{-1}(y)). \tag{1.13}$$

$(F(t,z))_{t\in\mathbb{C}}$ is, in its standard form,

$$F(t,z) = \tilde{g}_0(e^t \tilde{g}_0^{-1}(z)).$$

We also have a converse result for iteration groups, namely

Theorem 1.9. *Let $k \geq 1$ and let $F(t,z) = e^t z + \dots$ ($t \in \mathbb{C}$) be an analytic iteration group of type I with convergent infinitesimal generator $N_0(y) = \frac{\partial}{\partial t} F(t,y)|_{t=0}$. Then there exists a unique (GDh,φ) such that the set of non-zero local analytic solutions $\mathcal{L}_\varphi \setminus \{0\}$ can be written as*

$$f^\star(t,z) = F(t, \tilde{g}_0(z^k)) \qquad (t \in \mathbb{C}).$$

The generator $\tilde{g}_0$ of (GDh,φ) is given by the standard form $\tilde{g}_0(e^t \tilde{g}_0^{-1}(z))$, ($t \in \mathbb{C}$) of $(F(t,z))_{t\in\mathbb{C}}$.

Remark 1.10. $\mathcal{L}_\varphi \setminus \{0\}$ is an abelian group under the operation $\star$, defined by

$$(f_t^\star \star f_s^\star)(z) = f_t^\star((\tilde{g}_0^{-1}(f_s^\star(z)))^{\frac{1}{k}}) \qquad (t, s \in \mathbb{C}).$$

$(f_t^\star)_{t\in\mathbb{C}}$ satisfy a modified translation equation (see (1.16)).

1.1.3. *Representations of the solutions involving infinite products.* This section is devoted to solutions of the generalized Dhombres equation which can be represented by means of infinite products. We use the series

$$\tilde{X}(y) = -\sum_{\nu=0}^{\infty} \tilde{C}(y^{(k+1)^\nu}) \tag{1.10}$$

which was derived in the proof of Theorem 1.1 and Theorem 1.2, to obtain the following theorem.

Theorem 1.11. *Let $\varphi(y) = y^{k+1} + d_{k+2}y^{k+2} + \dots$ be holomorphic at $y = 0$, $k \geq 1$ (resp. formal).*

(1) *Then the general solution $f \in \mathcal{L}_\varphi \setminus \{0\}$ of (GDh,φ) is given by*

$$f(z) = B_\varphi^{-1}\left(\left(\left[uz \cdot \prod_{\nu=0}^{\infty} \frac{z^{k(k+1)^\nu}}{B_\varphi^{-1}(z^{k(k+1)^\nu})}\right]^{[-1]}\right)^k\right)$$

where $u \in \mathbb{C} \setminus \{0\}$ and ${}^{[-1]}$ denotes the substitutional inverse. These products converge in the order topology (if formal solutions are considered) or uniformly in each compact subset of a certain neighbourhood of 0.

(2) *Define* ψ *by* $\psi(z)^k = \varphi(z^k)$, $\psi(z) = z^{k+1} + \ldots$, *then the general solution in* $\mathcal{L}_\varphi \setminus \{0\}$ *is*

$$f(z) = \left(\left[uz \cdot \prod_{\nu=0}^{\infty} \frac{\psi^{\nu+1}(z)}{(\psi^\nu(z))^{k+1}}\right]^{[-1]}\right)^k$$

where $u \in \mathbb{C} \setminus \{0\}$ *and* ψ^ν *is the* ν*-th iterate of* ψ.

1.1.4. *Comparison of two* (GDh,φ). Let $k \geq 1$ and $\varphi_j(y) = y^{k+1} + d_{k+2}y^{k+2} + \ldots$, $j = 1, 2$, $B_{\varphi_1}, B_{\varphi_2}$ the Böttcher functions of φ_1 and φ_2. Denote by $U_j(u, z)$ the solution $U_j(u, z) = uz + \ldots$ of

$$\begin{cases} B_\varphi^{-1}(z^k)U(z) = U(z^{k+1}) \\ \operatorname{ord} U = 1 \end{cases} \tag{1.6}$$

for $\varphi = \varphi_j$, $j = 1, 2$. Then

$$B_{\varphi_1}^{-1}\left(\left(U_1(u,z)^{[-1]}\right)^k\right) \xrightarrow{\tau} B_{\varphi_2}^{-1}\left(\left(\prod_{\nu=0}^{\infty} \frac{B_{\varphi_1}^{-1}}{B_{\varphi_2}^{-1}}(z^{k(k+1)^\nu})U_1(u,z)^{[-1]}\right)^k\right) \qquad (u \in \mathbb{C}\setminus\{0\})$$

is a bijection from the set $\mathcal{L}_{\varphi_1} \setminus \{0\}$ to the set $\mathcal{L}_{\varphi_2} \setminus \{0\}$. Taking $\varphi_1(y) = y^{k+1}$, $U_1(u, z) = uz$ we obtain Theorem 1.11 (1).

1.2. $f(0) = w_0$, $w_0 \neq 0$, $w_0 \notin \mathbb{E}$. The Schröder case. The Schröder case is described in [16], almost all results presented in this section are taken from there. For the definition of a Siegel number the reader may consult [21].

Let us start with some transformations. From these transformations in this situation we have the necessary conditions:
If $f(z) = w_0 + g(z)$, $g(0) = 0$, ord $g = k \geq 1$, is a solution of (GDh,φ), then we have $\varphi(w_0) = w_0$,

$$\tilde{\varphi}(y) = w_0^k y + \ldots \tag{1.17}$$

with $\varphi(y) = w_0 + \tilde{\varphi}(y - w_0)$, and

$$g(z(w_0 + g(z))) = \tilde{\varphi}(g(z)) \tag{1.18}$$

formally or in a neighbourhood of $z = 0$.

Remark 1.12. If $w_0 \notin \mathbb{E}$, then k is uniquely determined from (1.17), furthermore

$$w_0 \notin \mathbb{E} \Leftrightarrow \omega_0 := w_0^k \notin \mathbb{E}.$$

The following theorem is well-known, that is why this section is called the Schröder case.

Theorem 1.13. *Let* $w_0 \in \mathbb{C} \setminus \mathbb{E}$, $\tilde{\varphi}(y) = w_0^k y + \ldots$. *Then there exist exactly one* $S_{\tilde{\varphi}}$, $S_{\tilde{\varphi}}(z) = z + \ldots \in \Gamma_1$ *such that*

$$\tilde{\varphi}(y) = S_{\tilde{\varphi}}^{-1}(w_0^k S_{\tilde{\varphi}}(y)). \tag{1.19}$$

$S_{\tilde{\varphi}}$ *is the Schröder function of* $\tilde{\varphi}$. *If* $\tilde{\varphi}$ *is local analytic,* $\left|w_0^k\right| \neq 1$ *or* w_0^k *is a Siegel number, then* $S_{\tilde{\varphi}}$ *is convergent.*

If we define $h := S_{\tilde{\varphi}} \circ g$, then

$$\text{(GDh,}\varphi) \;\Leftrightarrow\; \begin{cases} h(z(w_0 + S_{\tilde{\varphi}}^{-1}(h(z))) = w_0^k h(z) \\ \text{ord } h = k. \end{cases} \tag{1.20}$$

We have the following theorem which deals with formal solutions. Again a generator, which makes it possible to represent all solutions in a reasionable form, plays an important role.

Theorem 1.14 (Formal part)**.** *Let $w_0 \in \mathbb{C}$, $w_0 \neq 0$, $w_0 \neq \mathbb{E}$. Assume $\tilde{\varphi}(y) = w_0^k y + \dots$ (with $k \in \mathbb{N}$), $\varphi(w_0) = w_0$ Then the following holds true.*

1. *The set $\mathcal{L}_\varphi$ of all formal solutions f, $f(z) = w_0 + g(z)$, ord $g = k$, of* (GDh,φ) *can be described as follows:*
   There exists $\tilde{g}_0$, $\tilde{g}_0(y) = y + \dots$, uniquely determined by $\varphi(\tilde{\varphi})$ such that
   $$\mathcal{L}_\varphi = \left\{ w_0 + \tilde{g}_0(c_k z^k) \colon c_k \in \mathbb{C} \right\},$$
   where $\tilde{g}_0$ is the generator of (GDh,φ)*. To each $c_k \in \mathbb{C}$ there exists a unique solution f of* (GDh,φ) *with $f(z) = w_0 + c_k z^k + \dots$.*
2. *The generator $\tilde{g}_0$ of* (GDh,φ) *is given as unique solution $\tilde{g}_0$ with $\tilde{g}_0(y) = y + \dots$ of the functional equation*
   $$(w_0 + y)^k \tilde{g}_0^{-1}(y) = \tilde{g}_0^{-1}(y). \tag{1.21}$$
   We have
   $$\tilde{\varphi}(y) = \tilde{g}_0((w_0 + y)^k \tilde{g}_0^{-1}(y)). \tag{1.22}$$

The interested reader may compare this with (1.3) and (1.4), which are the settings in the Böttcher case. For local analytic solutions we obtain the following theorem.

Theorem 1.15 (Local analytic solutions)**.** *Let $w_0 \in \mathbb{C}$, $w_0 \neq 0$, $w_0 \neq \mathbb{E}$. Assume $\tilde{\varphi}(y) = w_0^k y + \dots$ (with $k \in \mathbb{N}$), $\varphi(w_0) = w_0$ and $\tilde{\varphi}$ convergent.*
If $|w_0| \neq 1$ or w_0 is a Siegel number, then all formal solutions of $\mathcal{L}_\varphi = \left\{ w_0 + \tilde{g}_0(c_k z^k) \colon c_k \in \mathbb{C} \right\}$ are convergent and yield local analytic solutions of (GDh,φ)*. In particular, $\tilde{g}_0$ is convergent.*

Remark 1.16. If $1 < |w_0| < 1$, then there exist representations of the solutions involving infinite products.
From (1.20) we come to the equation

$$\text{Ln}(1 + w_0^{-1} S_{\tilde{\varphi}}(y)) + \tilde{X}(y) = \tilde{X}(w_0 y) \qquad (\text{ord } \tilde{X} \geq 1)$$

where $S_{\tilde{\varphi}}$ is the Schröder function of $\tilde{\varphi}$.
If "iteration" of this linear equation is possible, we get with $\tilde{C}(y) = \text{Ln}(1 + w_0^{-1} S_{\tilde{\varphi}}(y))$

$$\tilde{X}(y) = -\sum_{\nu=0}^{\infty} \tilde{C}(w_0^\nu y). \tag{1.23}$$

The meaning of (1.23) is the following.
If $1 < |w_0| < 1$, then in the formal case the right hand side of (1.23) converges in the weak topology in $\mathbb{C}[\![y]\!]$ (coefficientwise), but not with respect to order; and if $\tilde{\varphi}$ is local analytic at 0, then the right hand side converges uniformly in each compact subset of a certain neighbourhood of 0. (1.23) leads to infinite products meaningful

in the respective topologies. The product representations can be found in [18], for a description of the weak topology we refer the reader to [26].

1.3. $f(0) = w_0 = 1$. This situation is investigated in [24]. Let $w_0 = 1$, then the function $\tilde{\varphi}$ is given by $\tilde{\varphi}(y) = 1^k y + \ldots = y + d_2 y^2 + \ldots$. We have the following theorem.

Theorem 1.17 ($w_0 = 1$)**.** (1) (GDh,φ) *has a non-constant formal solution* f *with* $f(0) = 1$, $f(z) = 1 + g(z)$, *ord* $g = k \in \mathbb{N}$ *if and only if*

$$\tilde{\varphi}(y) \equiv y + ky^2 \pmod{y^3}. \tag{1.24}$$

Hence k *is uniquely determined by* φ.

(2) *If* $\tilde{\varphi}$ *has the form* (1.24) *with* $k \in \mathbb{N}$, *then there exists a unique* $\tilde{g}_0$, $\tilde{g}_0(y) = y + \ldots$ *such that*

$$\mathcal{L}_\varphi = \left\{1 + \tilde{g}_0(c_k z^k) \colon c_k \in \mathbb{C}\right\}.$$

The existence of convergent (local analytic) solutions is open, nevertheless [24] contains one example of a convergent solution. Furthermore a partial answer to this is the following. There is a connection with covariant embeddings of a linear functional equation with respect to an iteration group, see [4] which is usefull to prove this theorem, namely we can apply the following idea.

Proving Theorem 1.17 we use the functional equation

$$\left[\frac{1}{k}\operatorname{Ln}\frac{y}{\tilde{\varphi}(y)} + \operatorname{Ln}(1+y)\right] + \tilde{X}(y) = \tilde{X}(\tilde{\varphi}(y))$$

with ord $\tilde{X} \geq 1$. Then we define

$$\tilde{C}(y) := \frac{1}{k}\operatorname{Ln}\frac{y}{\tilde{\varphi}(y)} + \operatorname{Ln}(1+y).$$

Let $(\phi_t(y))_{t\in\mathbb{C}}$ be the unique embedding of $\tilde{\varphi}$ into an analytic iteration group. Then there exists a unique $\Delta(t,y) \in \mathbb{C}[\![y]\!]$ with polynomials in t as coefficients such that

$$\begin{cases} \Delta(t,y) + \tilde{X}(y) = \tilde{X}(\phi_t(y)) & \text{for all } t \in \mathbb{C} \\ \Delta(1,y) = \tilde{C}(y),\ \Delta(0,y) = 0. \end{cases}$$

This system of equations yields a so called covariant embedding of the linear functional equation for $\tilde{X}$ (mentioned above) with respect to the analytic iteration group $(\phi_t(y))_{t\in\mathbb{C}}$. Let $H(y)$ be the infinitesimal generator of $(\phi_t)_{t\in\mathbb{C}}$ and define

$$\left.\frac{\partial}{\partial t}\Delta(t,y)\right|_{t=0} =: K(y).$$

Then

$$\tilde{X}(y) = \int_0^y \frac{K(y)}{H(y)}\,\mathrm{d}\,y.$$

Hence we have that $\tilde{g}_0$ is convergent, if K and H are convergent, K and H are given by φ.

1.4. $f(0) = w_0 \in \mathbb{E}$**, ord** w_0 **divides** k**.** This special case of the initial values of the generalized Dhombres equation is investigated in [22]. We see that we get a condition for the second coefficient of the function $\tilde{\varphi}$.

Let us assume that ord $w_0 = l \geq 2$. We define $\omega_0 := w_0^k$, then

$$\text{ord } \omega_0 = \frac{l}{\gcd(k, l)} =: l_0.$$

If ω_0 is given, then k is determined modulo l, $\gcd(k, l)$ does only depend on the residue class $k \pmod{l}$. For the following theorem we assume that ord $w_0 = l \geq 2$, $w_0^k = 1$ or $w_0 = 1$. This is equivalently to $k \equiv 0 \pmod{l}$, $k \in \mathbb{N}$. Then the theorem states as follows.

Theorem 1.18. *Assume ord $w_0 \geq 2$, $w_0^k = 1$ ($k \in \mathbb{N}$), $\tilde{\varphi}(y) = y + \ldots$.*

(1) (GDh,φ) *has a non-constant formal solution f with $f(0) = w_0$, $f(z) = w_0 + g(z)$, ord $g = k \in \mathbb{N}$ if and only if*

$$\tilde{\varphi}(y) \equiv y + w_0^{-1} k y^2 \pmod{y^3}. \tag{1.25}$$

(2) *If $\tilde{\varphi}$ has the form* (1.25) *with $k \in \mathbb{N}$, then there exists a unique $\tilde{g}_0$, $\tilde{g}_0(y) = y + \ldots$ such that*

$$\mathcal{L}_\varphi = \left\{ w_0 + \tilde{g}_0(c_k z^k) \colon c_k \in \mathbb{C} \right\}.$$

(3) *k is uniquely determined by φ.*

1.5. $f(0) = w_0 \in \mathbb{E}$**. The general case.** The general case is described in [22] and [23]. There we can find a complete characterization of the generalized Dhombres equations where the solutions take a root of one as value in 0. In this situation we have to distinguish the cases where $\tilde{\varphi}$ is linearizable and where it is not. For a linearizable $\tilde{\varphi}$ again the Schröder equation is used, otherwise if $\tilde{\varphi}$ is not linearizable semicanonical forms are needed, they are described in the following theorem. But before we consider this theorem, we want to specialise the setting of this section.

Let ord $w_0 = l \geq 2$. We define $\omega_0 := w_0^k$, then again ord $\omega_0 = \frac{l}{\gcd(k,l)} =: l_0$. Let $\tilde{\varphi}(y) = \omega_0 y + \ldots \in \mathbb{C}[\![y]\!] \cap \Gamma$ and let $l_0 \geq 2$. Then the following theorem on semicanonical forms holds true, for higher dimensions the reader may consult [20].

Theorem 1.19. *Let $\tilde{\varphi}$ be given by $\tilde{\varphi}(y) = \omega_0 y + \ldots \in \mathbb{C}[\![y]\!] \cap \Gamma$, ord $\omega_0 = l_0 \geq 2$.*

(1) *Then there exists a unique $R \in \Gamma_1$, $R(z) = y + \ldots$ such that*

$$R^{-1}(y) = y + \sum_{\substack{\nu \geq 1 \\ \nu \not\equiv 1 \pmod{k}}} \rho_\nu y^\nu$$

and

$$N(y) = (R \circ \tilde{\varphi} \circ R^{-1})(y) \in y\mathbb{C}[\![y^{l_0}]\!] \cap \Gamma,$$

that is $N(y) = \omega_0 y + \delta_{l_0+1} y^{l_0+1} + \ldots$, the semicanonical from of $\tilde{\varphi}$.

(2) *If $N(y) = \omega_0 y$, then $\tilde{\varphi}$ is linearizable, and each semicanonical form of $\tilde{\varphi}$ is linear.*

(3) *$N(y) \neq \omega_0 y$, then there exists $\nu_0 \in \mathbb{N}$ such that*

$$N(y) = \omega_0 y + \delta_{\nu_0 l_0 + 1} y^{\nu_0 l_0 + 1} + \ldots$$

where $\delta_{\nu_0 l_0+1} \neq 0$, ν_0 is the same for all semicanonial forms of $\tilde{\varphi}$.

1.5.1. $\tilde{\varphi}$ *is linearizable.* We start with the linearizable situation, hence let $\tilde{\varphi}$ be linearizable. Then according to Theorem 1.19 we have

$$\tilde{\varphi}(y) = R^{-1}(\omega_0 R(y))$$

where

$$R^{-1}(y) = y + \sum_{\substack{\nu \geq 1 \\ \nu \not\equiv 1 \pmod k}} \rho_\nu y^\nu \qquad (\rho_{\mu l_0 + 1} = 0, \mu \geq 1).$$

The formal solutions of the situation where $\tilde{\varphi}$ is linearizable are given by the following theorem.

Theorem 1.20 (Formal part)**.** *Let* $\operatorname{ord} w_0 = l \geq 2$, $\omega_0 := w_0^k$ *and* $\operatorname{ord} \omega_0 = \frac{l}{\gcd(k,l)} =: l_0 \geq 2$ *and let* $\tilde{\varphi}$ *be linearizable. Then the following holds true:*

(1) *If* (GDh,φ) *has a non-constant formal solution* f *with* $f(0) = w_0$, $f(z) = w_0 + g(z)$, $\operatorname{ord} g(z) = k^\star$ *for one* $k^\star$ *in the residue class* $k \pmod l$*, then it has such a solution for all members of this residue class.*

(2) *There exists a family of polynomials* $(Q_{\mu l_0})_{\mu \geq 1}$*, uniquely determined by* l_0*, with the following property:*
(GDh,φ) *has a non constant formal solution* f, $f(0) = w_0$, $f(z) = w_0 + g(z)$, $\operatorname{ord} g \in k \pmod l$ *if and only if*

$$\rho_{\mu l_0} = Q_{\mu l_0}(\rho_2, \dots, \rho_{l_0}, \rho_{l_0+2}, \dots, \rho_{\mu l_0 - 1}), \qquad \mu \geq 1. \tag{1.26}$$

(3) *If the system* (1.26) *is satisfied, then the general solution of* (GDh,φ) *with* $\tilde{\varphi}(y) = R^{-1}(\omega_0 R(y))$ *can be described as follows:*
Let $\tilde{X}_0$ *be the unique solution with* $\operatorname{ord} \tilde{X}_0 \geq 1$ *of*

$$\begin{cases} \operatorname{Ln}(1 + w_0^{-1} R^{-1}(y)) + \tilde{X}_0(y) = \tilde{X}_0(\omega_0 y), \\ \tilde{X}_0(y) = \displaystyle\sum_{\nu \not\equiv 0 \pmod{l_0}} \tilde{\xi}_\nu y^\nu, \end{cases}$$

where $\operatorname{Ln}(1 + w_0^{-1} R^{-1}(y)) = \sum_{\nu \not\equiv 0 \pmod{l_0}} \theta_\nu y^\nu$*, then the general solution is given by*

$$w_0 + R^{-1}\left(\left(\left[uz \exp \tilde{X}_0(z^k) \cdot \Theta(z)\right]^{[-1]}\right)^k\right), \qquad u \in \mathbb{C} \setminus \{0\} \tag{1.27}$$

where $\Theta(z)$ *is an arbitrary series of the form* $\Theta(z) = 1 + \sum_{\nu \geq 1} \theta_{\nu l_0} z^{\nu l_0} \in \mathbb{C}[\![z^{l_0}]\!]$*. This representation is unique.*

Remark 1.21. (1) In the situation of Theorem 1.20 we have as necessary and sufficient conditions for the existence of non-constant formal solutions an <u>infinite</u> system of algebraic relations, namely (1.26).

(2) If (1.26) is satisfied we have a set of solutions which depends on the <u>arbitrary</u> function $\Theta(z) = 1 + \ldots \in \mathbb{C}[\![z^{l_0}]\!]$ (see (1.27)).

(3) If we choose $\Theta = 1$, then we obtain a subset of solutions which can also be given by a generator $\tilde{g}_0$, $\tilde{g}_0(y) = y + \ldots$ as $\{w_0 + \tilde{g}_0(c_k z^k) \colon c_k \in \mathbb{C}\}$. ($\tilde{g}_0$ can be seen as a "<u>subgenerator</u>" of $\mathcal{L}_\varphi$, it is valid for each k from the residue class $k \pmod l$). This is not the only subset of $\mathcal{L}_\varphi$ which has a generator.

Let $\Theta(z) = 1 + \ldots \in \mathbb{C}[\![z^l]\!] \cap \mathbb{C}[\![z^k]\!]$, then the set of solutions

$$w_0 + R^{-1}\left(\left(\left[uz \exp \tilde{X}_0(z^k) \cdot \Theta(z)\right]^{[-1]}\right)^k\right), \qquad u \in \mathbb{C} \setminus \{0\} \tag{1.28}$$

has also a subgenerator.

(4) The system (1.26) can always be satisfied, since ρ_ν with $\nu \not\equiv 0 \pmod{l_0}$, $\nu \not\equiv 1 \pmod{l_0}$ can be chosen arbitrarily. Hence there exists a continuum of admissible $\tilde{\varphi}(y) = R^{-1}(\omega_0 R(y))$.

For the convergence of the solutions we need the following (see [19]):
Let $\tilde{\varphi}(y) = R^{-1}(\omega_0 R(y))$ be convergent, where $R^{-1}(y) = y + \sum_{\substack{\nu \geq 1 \\ \nu \not\equiv 1 \pmod{k}}} \rho_\nu y^\nu \in \Gamma_1$. Then R^{-1} (and also R) may be chosen convergent. This leads to

Theorem 1.22 (Local analytic solutions)**.** *Let* $\operatorname{ord} w_0 = l \geq 2$*,* $\omega_0 := w_0^k$ *and* $\operatorname{ord} \omega_0 = \frac{l}{\gcd(k,l)} =: l_0 \geq 2$ *and let* $\tilde{\varphi}$ *be linearizable and convergent. Let* $\tilde{\varphi}$ *(that is* R^{-1}*) satisfy* (1.26) *and let* R^{-1} *be convergent (which is possible).*

(1) *Then* $\tilde{X}_0(z^k)$ *from Theorem 1.20 (3) is convergent, hence all solutions*

$$w_0 + R^{-1}\left(\left(\left[uz \exp \tilde{X}_0(z^k) \cdot \Theta(z)\right]^{[-1]}\right)^k\right)$$

are local analytic.

(2) *A solution of the form* (1.27) *is local analytic if and only if* Θ *is convergent.*

We can ask whether it is possible to find $\tilde{\varphi}(y) = R^{-1}(\omega_0 R(y))$ which is convergent and such that R satisfies (1.26)? This gives a class of examples, see [22] and [23].

Example 1.23. Let $w_0 \in \mathbb{E}$, $\operatorname{ord} w_0 = l \geq 2$, $k \in \mathbb{N}$, $\omega_0 = w_0^k$, $\operatorname{ord} \omega_0 = l_0 \geq 2$. Then

$$\tilde{\varphi}(y) = \frac{\omega_0 y}{1 + \beta y}$$

can be linearized, as a Möbius transformation and as a formal series. The following holds true:
There are finitely many values of β such that (GDh,φ) has a non-constant solution. For these values of β and each $\bar{k}$ from $k \pmod{l}$ one has explicit representations of local analytic solutions of (GDh,φ), using certain binomial expressions. (The proof works directly without using Theorem 1.22).

1.5.2. *$\tilde{\varphi}$ is not linearizable.* Let $\tilde{\varphi}$ be not linearizable. We assume again that $\operatorname{ord} w_0 = l \geq 2$, $\omega_0 := w_0^k$, $k \in \mathbb{N}$ and $\operatorname{ord} \omega_0 = \frac{l}{\gcd(k,l)} =: l_0 \geq 2$. By Theorem 1.19 we write

$$\begin{aligned} \tilde{\varphi}(y) &= R^{-1}(\omega_0 R(y)) \\ R^{-1}(y) &= y + \sum_{\substack{\nu \geq 1 \\ \nu \not\equiv 1 \pmod{k}}} \rho_\nu y^\nu. \end{aligned} \tag{1.29}$$

The function $N(y)$ is given by

$$N(y) = \omega_0 y + \sum_{\nu \geq 1} \delta_{\nu l_0 + 1} y^{\nu l_0 + 1} \tag{1.30}$$

where $N(y) \neq \omega_0 y$, i.e. there exists a $\nu_0 \in \mathbb{N}$ such that

$$N(y) - \omega_0 y = \delta_{\nu l_0+1} y^{\nu l_0+1} + \ldots, \qquad \delta_{\nu l_0+1} \neq 0.$$

Then we have the next theorem.

Theorem 1.24. *Let w_0, k, ω_0, $\nu_0 \geq 1$ be given such that ord $w_0 = l \geq 2$, $\omega_0 := w_0^k$ and ord $\omega_0 = l_0 \geq 2$ and assume that $\tilde{\varphi}(y) = R^{-1}(\omega_0 R(y))$, where R^{-1} and N satisfy* (1.29) *and* (1.30). *Then the following holds:*

(1) *There exists a family of polynomials $P_{l_0}, P_{2l_0}, \ldots, P_{(\nu_0-1)l_0}, P_{\nu_0 l_0}$ (only depending on w_0, ω_0, ν_0) with the following property:*
(GDh,φ) *has a non-constant formal solution f with $f(0) = w_0$ if and only if*

$$\begin{aligned} \rho_{l_0} = P_{l_0}(\rho_2, \ldots, \rho_{l_0-1}), \ldots, \rho_{(\nu_0-1)l_0} &= P_{(\nu_0-1)l_0}(\rho_2, \ldots, \rho_{(\nu_0-1)l_0-1}) \\ 0 = -\delta_{\nu_0 l_0+1} + k(\rho_{\nu_0 l_0} &- P_{\nu_0 l_0}(\rho_2, \ldots, \rho_{\nu_0 l_0-1})) \end{aligned}$$

and

$$\rho_{\nu_0 l_0} \neq P_{\nu_0 l_0}(\rho_2, \ldots, \rho_{\nu_0 l_0-1}). \tag{1.31}$$

(2) *If these conditions are fulfilled, then there exists a unique $\tilde{g}_0$, $\tilde{g}_0(y) = y + \ldots$ such that the set $\mathcal{L}_\varphi$ of all formal solutions f of* (GDh,φ) *is given by* $\{w_0 + \tilde{g}_0(c_k z^k) \colon c_k \in \mathbb{C}\}$.
(3) *For a given function $\tilde{\varphi}$ only one value of k, given by* (1.31), *is possible.*

Remark 1.25. (1) The second line in (1.31) shows that there exists at most one admissible k. Also $k \pmod{l}$ is known from φ.
(2) The conditions (1.31) are easy to fulfill. The coefficients $\delta_{\nu l_0+1}$, for $\nu > \nu_0$ and ρ_μ for $\mu > \nu_0 l_0$, $\mu \not\equiv 1 \pmod{l_0}$ can be chosen arbitrarily. Hence there are local analytic $\tilde{\varphi}$ which satisfy (1.31), but it is open when there are formal solutions which converge.

We want to finish this section by giving a condition on $\tilde{\varphi}$ which indicates when there are solutions of (GDh,φ) exist. Therefore the following theorem deals with the solvability of a generalized Dhombres equation.

Theorem 1.26 (A general characterization of solvable (GDh,φ), $f(0) = w_0$)**.** *Let $\varphi(y) = d_0 + d_1(y - w_0) + d_2(y - w_0)^2 + \ldots \in \mathbb{C}[\![y - w_0]\!]$ ($\tilde{\varphi}(y) = d_1 y + d_2 y^2 + \ldots$). Then the following holds:*
(GDh,φ) *has a non-constant formal solution if and only if there exists w_0, $k \in \mathbb{N}$ and a series $\tilde{g}_0$, $\tilde{g}_0(y) = y + \ldots$ such that*

$$\tilde{\varphi}(y) = \tilde{g}_0(\tilde{g}_0^{-1}(y)(w_0 + y)^k).$$

Remark 1.27. (1) It would be desirable to prove Theorem 1.26 without using the whole theory of generalized Dhombres equations.
(2) We want to mention that there are also results for the connection of generalized Dhombres equations where $f(0) = w_0 \neq 0$ with Briot-Bouquet differential equations and iteration groups of type I.

1.6. $f(z_0) = 1$**,** $z_0 \neq 0$**.** Briefly we want to present the case where $f(z_0) = 1$ and $z_0 \neq 0$, details can be found in [22]. We can write

$$f(z) = 1 + \hat{g}(z), \qquad \text{ord}\,\big|_{z=1}\,\hat{g} = k \geq 1$$

and we set $\hat{f}(z) = g(\zeta)$ where $z = 1 + \zeta$. We obtain

$$(1.32) \qquad (\mathrm{GDh},\varphi) \ \Leftrightarrow\ g(\zeta + g(\zeta) + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)), \qquad \operatorname{ord} g = k$$

where $\varphi(y) = 1+\tilde{\varphi}(y-1)$, $\tilde{\varphi}(0) = 0$. Equation (1.32) replaces equation (1.18) (there we have $f(0) = w_0$, or 0). We want to summarize this situation in the following remark.

Remark 1.28.
(1) Only $k = 1$ is possible. (All non-constant local analytic or formal solutions are invertible).
(2) There exist φ ($\tilde{\varphi}$)for which only one non-constant formal solution exists.
(3) There exists $\tilde{\varphi}$ for which the set of all formal solutions is given by a generator,
$$f(z) = 1 + \tilde{g}_0(c_1(z-1)), \qquad \tilde{g}_0(y) = y + \dots .$$
(4) There exist φ for which the set of all formal solutions depends on an "arbitrary function and has a "subgenerator".
(5) There are "Schröder" cases and "Böttcher" cases.
(6) A sequence of transformations leads from (1.32) to
$$\begin{cases} (1+\zeta)V(\zeta) = V(\tilde{\varphi}(\zeta)) \\ V(\zeta) = 1 + \dots, \end{cases}$$
which does not depent on z_0 explicitly. The factor of V on the left hand side does not depend on $\tilde{\varphi}$, for example compare this with the case $f(0) = w_0$ where an equation
$$\frac{z(w_0 + z^k)}{\psi(z)} V(z) = V(\psi(z))$$
with $\psi(z)^k = \varphi(z^k)$ appears. We see, that for $k = 1$ we have
$$\frac{z(w_0 + z)}{\tilde{\varphi}(z)} V(z) = V(\tilde{\varphi}(z)).$$

2. Some global results

In this section we present results concerning non-constant polynomial or rational solutions of the generalized Dhombres equation. The local and formal theory, described above, leads in combination with some basic results from polynomial algebra and field theory, as well as from complex analysis to a characterization of these equations. Furthermore the equation $f(zf(z)) = f(z)^{k+1}$ is considered on certain regions.

2.1. **The polynomial case.** This section deals with the polynomial solutions but also entire solutions are discussed. The reader my consult [13]. We have the following theorem.

Theorem 2.1. *Assume that φ is an entire function and f_0 a non-constant complex polynomial such that*

$$(\mathrm{GDh},\varphi) \qquad f_0(zf_0(z)) = \varphi(f_0(z)), \qquad z \in \mathbb{C}$$

holds. Denote $f_0(0)$ by w_0.

(1) *Then there exists* $k \in \mathbb{N}$ *such that*

$$\varphi(y) = w_0 + (y - w_0)y^k, \qquad y \in \mathbb{C} \tag{2.1}$$

and

$$f_0(z) = w_0 + c_k^{(0)} z^k, \qquad z \in \mathbb{C} \tag{2.2}$$

with $c_k^{(0)} \neq 0$.

(2) *If* φ *is given by* (2.1)*, then the general formal (local analytic) solution of* (GDh,φ) *is given by*

$$f(z) = w_0 + c_k z^k, \qquad c_k \in \mathbb{C}.$$

For $w_0 = 0$ we get by Theorem 2.1 a characterization of the equations

$$f(zf(z)) = f(z)^{k+1} \qquad (k \geq 1)$$

which where studied by J. Smítal and P. Kahlig in the real domain. Theorem 2.1 has the following converse.

Theorem 2.2. *If* $w_0 \in \mathbb{C}$, $k \in \mathbb{N}$, $\varphi(y) = w_0 + (y - w_0)y^k$, *(*$y \in \mathbb{C}$*), then* (GDh,$\varphi$) *has, on* $\mathbb{C}$*, the non-constant polynomial solutions* $f(z) = w_0 + c_k z^k$, $c_k \in \mathbb{C}$*, which are all formal (local analytic) solutions.*

What can be said about entire solutions of (GDh,φ) which are not polynomial, that means we are investigating equations

$$f(zf(z)) = \varphi(f(z)), \qquad z \in \mathbb{C} \tag{GDh,φ}$$

where φ is an entire function and f is an transcendental entire one. We have the following remark, see page 308 of [8].

Remark 2.3 (M. Laczkovich). Let φ be an entire function, then (GDh,φ) cannot have an entire transcendental solution.

2.2. **The rational case.** In this section we characterize those generalized Dhombres equations (GDh,φ) having non-constant rational solutions which are holomorphic at 0. These results are taken from [17]. We have

Theorem 2.4. *Let* φ *be a meromorphic function, which is holomorphic at* $w_0 \in \mathbb{C}$*. Let* f_0 *be a non-constant rational function, holomorphic at* 0*, with* $f(0) = w_0$*. Assume that*

$$f_0(zf_0(z)) = \varphi(f_0(z))$$

holds for $|z| < R$*. Then* (φ, f_0) *fulfills one of the following conditions:*

(1) *There exist* $k \in \mathbb{N}$, $\beta_0 \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C} \setminus \{0\}$ *such that*

$$\varphi(w) = w_0 + \frac{(w - w_0)w^k}{\gamma(w - w_0)w^k - \gamma(w - w_0) + 1} \tag{2.3}$$

$$f_0(z) = w_0 + \frac{z^k}{\beta_0 + \gamma z^k}. \tag{2.4}$$

By analytic continuation, (φ, f_0) *fulfills* (GDh,φ) *in* $\mathbb{C} \cup \{\infty\}$.

(2) *There exist* $k \in \mathbb{N}$, $\beta \in \mathbb{C} \setminus \{0\}$ *such that*

$$\varphi(w) = w_0 + (w - w_0)w^k \tag{2.5}$$

$$f_0(z) = w_0 + \beta_0 z^k. \tag{2.6}$$

By analytic continuation, (φ, f_0) *fulfills* (GDh,φ) *in* $\mathbb{C} \cup \{\infty\}$.

(3) *If* $\beta_0 \in \mathbb{C} \setminus \{0\}$*, then* f *defined by*

$$f(z) = w_0 + \frac{z^k}{\beta + \gamma z^k} \tag{2.7}$$

is a solution of (GDh,φ) *with* φ *definded by* (2.3)*.*

(4) *If* $\beta_0 \in \mathbb{C} \setminus \{0\}$*, then* f *defined by*

$$f(z) = w_0 + \beta z^k \tag{2.8}$$

is a solution of (GDh,φ) *with* φ *definded by* (2.5)*.*

In Theorem 2.4 (2) we get, under different assumptions, the same functional equation and the same solutions we already had in Theorem 2.1 and Theorem 2.2. We also have a converse result.

Theorem 2.5. *Let* $w_0 \in \mathbb{C}$*,* $k \in \mathbb{N}$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$*. Then the following holds.*

(1) *For each* $\beta \in \mathbb{C} \setminus \{0\}$*,* f *defined by* (2.7) *is a non-constant rational solution of* (GDh,φ) *with* φ *defined by* (2.3)*, in* $\mathbb{C} \cup \{\infty\}$*.*

(2) *For each* $\beta \in \mathbb{C} \setminus \{0\}$*,* f *defined by* (2.8) *is a non-constant rational solution of* (GDh,φ) *with* φ *defined by* (2.5)*, in* $\mathbb{C} \cup \{\infty\}$*.*

The Taylor expansions of (2.7) and (2.8) at $z = 0$ are local analytic (formal) solutions of the corresponding (GDh,φ) with $f(0) = w_0$. Are there other local analytic or formal solutions? (Note that this is not the case in the situation of Theorem 2.1 and Theorem 2.2). Therefore we have

Theorem 2.6. (1) *Assume that* (φ, f_0) *fulfills the hypothesis of Theorem 2.4, and assume that* φ *is not the Möbius transformation*

$$\varphi(w) = w_0(1 + w_0 - w)^{-1},$$

where w_0 *is a root of one with ord* $w_0 \geq 2$*. Then all formal solutions* F *of* (GDh,φ)*,* $F(z) = w_0 + c_m z^m + \dots$ *are given as Taylor expansions of the solutions* (2.4) *and* (2.6) *at* $z = 0$*.*

(2) *If* $\varphi(w) = w_0(1 + w_0 - w)^{-1}$*,* w_0 *being a root of one, ord* $w_0 \geq 2$*, then* (GDh,φ) *has in addition to the rational solutions given by Theorem 2.4 also local analytic solutions and non convergent formal solutions which are not Taylor expansions of rational functions at* 0*.*

Remark 2.7. (1) By direct considerations one can also construct <u>all rational</u> functions φ for which (GDh,φ) has rational solutions having a <u>pole</u> at 0. (Example: $f(z) = \frac{1}{z}$ satisfies $f(zf(z)) = 1$ in $\mathbb{C} \cup \{\infty\}$).

(2) It should be possible to find all rational functions φ for which (GDh,φ) has a non-constant rational solution from the local theory for the case "$f(z_0) = 1$, $z_0 \neq 0$", maybe together with the local theory for the case "$f(0) = w_0$".

2.3. **The equation** $f(zf(z)) = f(z)^{k+1}$**.** We now come back to the equation

$$f(zf(z)) = f(z)^{k+1} \qquad (k \geq 1)$$

which is investigated in [15]. For this equation there are several results in various regions. These regions are

$$\mathbb{C} \setminus \{0\}\,,\ \{z\colon 0 < |z| < R_0\}\,,\ \{z\colon |z| > R_0\}\,,\ \{z\colon 0 < R_1 < |z| < R_2\}\,.$$

We have the following theorem.

Theorem 2.8. *Let $\varphi(y) = c_0 + \ldots + c_m y^m$ be a polynomial of degree m, $m \geq 1$. Let f be holomorphic in $\mathbb{C} \setminus \{0\}$ and assume that* (GDh,φ) *holds in $\mathbb{C} \setminus \{0\}$. then f can be analytically continued to $z = 0$ and remains a solution of* (GDh,φ) *on $\mathbb{C}$. (By the remark of Laczkovich and Theorem 2.2: If f is not constant, φ is of the special form* (2.5) *and f of the form* (2.6)*.)*

Furthermore we have

Theorem 2.9. *Let $k \in \mathbb{N}$, $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be holomorphic with $f(z) \neq 0$ for $z \in \mathbb{C} \setminus \{0\}$. Assume that f satisfies the functional equation*

$$f(zf(z)) = f(z)^{k+1} \qquad \text{for } z \in \mathbb{C} \setminus \{0\}\,.$$

Then either $f(z) = cz^k$ ($z \in \mathbb{C} \setminus \{0\}$), $c \neq 0$ or $f(z) = \zeta^m$ ($z \in \mathbb{C} \setminus \{0\}$) with $\zeta = e^{\frac{2\pi i}{k}}$, $m = 0, \ldots, k-1$.

All the solutions described in the following theorems can be analytically continued to $\mathbb{C}$ where they remain solutions.

Theorem 2.10. *Let $G = \{z\colon 0 < |z| < R_0\}$ with $R_0 > 0$, and let $f\colon G \to \mathbb{C}$ be holomorphic in G. Assume that*

$$f(zf(z)) = f(z)^{k+1} \qquad (z \in G)$$

holds, with $k \geq 1$.
Then either $f(z) = cz^k$, $(z \in G)$ with

$$|c| < \frac{1}{R_0^k}$$

or $f(z) = \zeta^m$, $(z \in G)$, with $\zeta = e^{\frac{2\pi i}{k}}$, $m = 0, \ldots, k-1$.
Conversely, $f\colon z \mapsto cz^k$ is for each c with $|c| < \frac{1}{R_0^k}$ a solution of our equation in G.

Theorem 2.11. *Let $R_0 > 0$, $G = \{z\colon |z| > R_0\}$. Assume that $f\colon G \to \mathbb{C}$ be holomorphic in G and satisfies*

$$f(zf(z)) = f(z)^{k+1} \qquad (z \in G).$$

Then $f(z) = cz^k$ with $|c| > \frac{1}{R_0^k}$, or $f(z) = \zeta^m$, $(z \in G)$, with $\zeta = e^{\frac{2\pi i}{k}}$, $m = 0, \ldots, k-1$.
Conversely, if $|c| > \frac{1}{R_0^k}$, then $f\colon z \mapsto cz^k$ is a solution of our equation in G.

Theorem 2.12. *Let G be a region in $\{z\colon 0 < R_1 < |z| < R_2\}$ where $0 < R_1 < R_2$. Let f be holomorphic in G and satisfy*

$$f(zf(z)) = f(z)^{k+1} \qquad (z \in G).$$

Then f is constant ($f = \zeta^m$).

3. Open problems

In this section we want to mention some open problems related to the generalized Dhombres equation in the complex domain. These problems are the following:

(1) The investigation of the existence of local analytic solutions f of (GDh,φ) with $f(z_0) = w_0$ for general z_0, w_0.
(2) If $w_0 \in \mathbb{E}$, then one has to investigate the convergence of the formal solutions for a convergent, not linearizable $\tilde{\varphi}$.

(3) One could also describe the dependence of the solutions on the initial values w_0.
(4) Find a class of iterative functional equations which is invariant under a group of local changes of coordinates which contains the generalized Dhombres equations.
(5) Find a connection between iterative functional equations and analytic iteration groups $(F_t(X))_{t\in\mathbb{C}}$ of type II (i.e. $F_t(X) = X + c_k(t)X^k + \ldots$, $t \in \mathbb{C}$, $k \geq 2$, $c_k \neq 0$) analoguous to the relation between (GDh,φ) and iteration groups of type I described in Theorem 1.8 and Theorem 1.9.

References

[1] H. Cartan, *Elementare Theorien der Analytischen Funktionen einer oder mehrerer Komplexen Veränderlichen*, Bibliographisches Institut, Mannheim/Wien/Zürich, (1966).
[2] J. Dhombres, *Applications associatives ou commutatives*, C. R. Acad. Sci. Paris, Sr. A **281**, 809-812 (1975).
[3] J. Dhombres, *Some Aspects of Functional Equations*, Chulalongkorn University Press, Bangkok, (1979).
[4] H. Fripertinger, L. Reich, *On covariant embeddings of a linear functional equation with respect to an analytic iteration group*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **13**, No. 7, 1853–1875 (2003).
[5] P. Henrici, *Applied and Computational Complex Analysis, Volume 1*, Wiley, (1988).
[6] E. Hille, *Ordinary differential equations in the complex domain. Unabr. and unaltered republ. of the 1976 ed.*, Dover, Mineola, NY, (1997).
[7] M. Kuczma, B. Choczewski, R. Ger, *Iterative Functional Equations*, Encyclopedia Math. Appl. **32**, Cambridge University Press, Cambridge (1994).
[8] M. Laczkovich, *A note on the Dhombres equation*, In: Report of Meeting - The Forty-ninth International Symposium on Functional Equations, June 19–26, 2011, Graz-Mariatrost, Austria, Aequationes Math. **84**, 285–318 (2012).
[9] L. Reich, *On a differential equation arising in iteration theory in rings of formal power series in one variable*, Iteration theory and its functional equations (Lochau, 1984), 135-148, Lecture Notes in Math. **1163**, Springer, Berlin, (1985).
[10] L. Reich, *Über verallgemeinerte Differentialgleichungen von Aczél-Jabotinsky im Komplexen. (Parameterabhängigkeit und holomorphe Fortsetzung von Integralen).*, Math. Pannonica **1**, No.1, 33–47 (1990).
[11] L. Reich, *Generalized Böttcher equations in the complex domain*, In: Aulaskari, Rauno (ed.) et al., Symposium on complex differential and functional equations. Papers from the symposium in honor of Ilpo Laine's 60th birthday, Joensuu, Finland, October 1415, 2002. Joensuu: University of Joensuu, Department of Mathematics. Report Series. Department of Mathematics, University of Joensuu 6, 135–147 (2004).
[12] L. Reich, J. Smítal, *Functional equation of Dhombres type a simple equation with many open problems*, J. Difference Equ. Appl. **15**, No. 11-12, 1179–1191 (2009).
[13] L. Reich, J. Smítal, *On generalized Dhombres equations with nonconstant polynomial solutions in the complex plane*, Aequationes Math. **80**, No. 1-2, 201–208 (2010).
[14] L. Reich, J. Smítal, M. Štefánková, *Local Analytic Solutions of the Generalized Dhombres Functional Equation I*, Sitzungsber. österr. Akad. Wiss. Wien, Math.-nat Kl. Abt. II **214**, 3–25 (2005).
[15] L. Reich, J. Smítal, M. Štefánková *The holomorphic solutions of the generalized Dhombres functional equation*, J. Math. Anal. Appl. **333**, No. 2, 880–888 (2007).
[16] L. Reich, J. Smítal, M. Štefánková, *Local Analytic Solutions of the Generalized Dhombres Functional Equation II*, J. Math. Anal. Appl. **355**, No. 2, 821–829 (2009).
[17] L. Reich, J. Smítal, M. Štefánková, *On generalized Dhombres equations with non-constant rational solutions in the complex plane*, J. Math. Anal. Appl. **399**, No. 2, 542–550 (2013).
[18] L. Reich, J. Tomaschek, *Some Remarks to the Formal and Local Theory of the Generalized Dhombres Functional Equation*, Results. Math. **63**, No. 1-2, 377–395 (2013).
[19] L. Reich, J. Tomaschek, *A remark on Schröder's equation: Formal and analytic linearization of iterative roots of the power series $f(z) = z$*, in preparation.

[20] J. Schwaiger, L. Reich, *Über die Funktionalgleichung $N \circ T = T \circ M$ für formale Potenzreihen*, Aequationes Math. **19**, 66–78 (1979).
[21] C. L. Siegel, *Vorlesungen über Himmelsmechanik*, Springer, Berlin (1956).
[22] J. Tomaschek, *Contributions to the local theory of generalized Dhombres functional equations in the complex domain*, Grazer Math. Ber. **358**, p. 72+iv (2011).
[23] J. Tomaschek, *Some Aspects of the local theory of generalized Dhombres functional equations in the complex domain*, ESAIM: Proc. **36**, European Conference on Iteration Theory 2010, 1–14 (2012).
[24] J. Tomaschek, L. Reich, *Formal solutions of the generalized Dhombres functional equation with value one at zero*, Aequationes Math. **83**, No. 1-2, 117-126 (2012).
[25] J. Tomaschek, L. Reich, *Local solutions of the generalized Dhombres functional equation in a neighbourhood of infinity*, submitted.
[26] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York-London (1967).

Institute for Mathematics and Scientific Computing, Karl-Franzens-University Graz, Heinrichstrasse 36, 8010 Graz, Austria, email: ludwig.reich@uni-graz.at

Mathematics Research Unit, University of Luxembourg, 6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg, Grand Duchy of Luxembourg, email: joerg.tomaschek@uni.lu